De Dirichlet a Esperanzas ¿O al revés?

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Funciones Multiplicativas

Definición 1

Una funcion $f : \mathbb{Z}^+ \to \mathbb{C}$ es llamada Multiplicativa si para cualesquiera $(n, m) = 1$

$$f(nm) = f(n)f(m)$$

Y es llamada Completamente Multiplicativa si:

$$f(nm) = f(n)f(m), \quad \forall n, m \in \mathbb{Z}$$

Ejemplos:

- $1(n) = 1$
- $\mathrm{id}_k(n) = n^k$
- $f(n) = \sqrt{n}$
- $\varphi(n) = \#\{m \leq n : (n, m) = 1\}$
- $\sigma_k(n) = \sum_{d|n} d^k$

Razon: Si $d \mid nm$ con $(n, m) = 1 \quad \Rightarrow \quad d \mid n$ ó $d \mid m$

- $\mu(n) = \begin{cases} (-1)^k & \text{si } n = p_1 \cdots p_k \\ 1 & \text{si } n = 1 \\ 0 & \text{e.o.c} \end{cases}$

Funciones Multiplicativas

Proposición 1

Si $f : \mathbb{Z}^+ \to \mathbb{C}$ es multiplicativa entonces $f(1) = 1$

Proposición 2

$f : \mathbb{Z}^+ \to \mathbb{C}$ es multiplicativa si, y solo si, $f(n) = \prod_{i=1}^{k} f(p_i^{k_i}) = \prod_p f(p^{\upsilon_p(n)})$

Proposición 3

$f : \mathbb{Z}^+ \to \mathbb{C}$ es completamente multiplicativa si, y solo si, $f(n) = \prod_p f(p)^{\upsilon_p(n)}$

Funciones Multiplicativas

Corolario

Se tiene que:

- $\sigma_0(n) = \prod_p (1 + \upsilon_p(n))$
- $\sigma_k(n) = \prod_p \dfrac{p^{(\upsilon_p(n)+1)k} - 1}{p^k - 1}$
- $\varphi(n) = \prod_p p^{\upsilon_p(n)}(1 - 1/p)$

Demostración. –

- $\sigma_0(n) = \prod_p \sigma_0(p^{\upsilon_p(n)}) = \prod_p (1 + \upsilon_p(n))$

Convolución de Dirichlet

Definición 2

Sean $f, g : \mathbb{Z}^+ \to \mathbb{C}$ se define a la Convolución de f y g como:

$$(f * g)(n) := \sum_{d|n} f(d) g(\tfrac{n}{d})$$

Ejemplos:

- $(1 * 1)(n) = \sum_{d|n} 1(d) \cdot 1(\tfrac{n}{d}) = \sum_{d|n} 1 = \sigma_0(n) \quad \Rightarrow \quad 1 * 1 = \sigma_0$
- $(\mathrm{id}_k * 1)(n) = \sum_{d|n} \mathrm{id}_k(d) \cdot 1(\tfrac{n}{d}) = \sum_{d|n} d^k = \sigma_k(n) \Rightarrow \mathrm{id}_k * 1 = \sigma_k$
- $(\varphi * 1)(n) = \sum_{d|n} \varphi(d) \cdot 1(\tfrac{n}{d}) = \sum_{d|n} \varphi(d) = n \quad \Rightarrow \quad \varphi * 1 = \mathrm{id}$

$\mathrm{id} * 1 = \sigma_1$

$\Rightarrow \ (\varphi * 1) * 1 = \sigma_1 \ \Rightarrow \ \varphi * (1 * 1) = \sigma_1 \Rightarrow \ \varphi * \sigma_0 = \sigma_1 \ \Rightarrow \ \sum_{d|n} \varphi(d) \sigma_0(\tfrac{n}{d}) = \sigma_1(n)$

Convolución de Dirichlet

Teorema 1

Si F denota el conjunto de funciones aritmeticas con $f(1) \neq 0$, entonces $(F, *)$ es un grupo abeliano.

Corolario

Si $\hat{F}$ denota el conjunto de funciones multiplicativas, entonces $(\hat{F}, *) \leq (F, *)$

¿Quien es el neutro?

El neutro de la operacion es: $\varepsilon(n) := \mathbb{I}_{n=1} \quad \Rightarrow \quad f * \varepsilon = f$

¿Inversos?

Formula de inversion de Möbius:

$$1 * \mu = \varepsilon \qquad \text{es decir} \qquad \mu = (1)^{-1}$$

Series de Dirichlet

Definición 3

Sean $f : \mathbb{Z}^+ \to \mathbb{C}$, se define la Serie de Dirichlet asociada a f como:

$$D_f(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \quad s \in \mathbb{C}$$

Obs.-

- Hay teoremas que garantizan la existencia de un semiplano de convergencia
- En tal caso la funcion $D_f(s)$ resulta analitica en dicho semiplano.

Ejemplos:

- $D_1(s) = \zeta(s)$
- $D_{\mathrm{id}}(s) = \sum_{n=0}^{\infty} \frac{\mathrm{id}(n)}{n^s} = \sum_{n=0}^{\infty} \frac{n}{n^s} = \sum_{n=0}^{\infty} \frac{1}{n^{s-1}} = \zeta(s-1)$
- $D_{\mathrm{id}_k}(s) = \sum_{n=0}^{\infty} \frac{\mathrm{id}_k(n)}{n^s} = \sum_{n=0}^{\infty} \frac{n^k}{n^s} = \zeta(s-k)$

Series de Dirichlet

¿Y esto de que me importa?

De forma directa tenemos:

$$\mathbb{E}[f(N_s)] = \sum_{n=1}^{\infty} f(n)\frac{1}{\zeta(s)}\frac{1}{n^s} = \frac{1}{\zeta(s)}\sum_{n=1}^{\infty}\frac{f(n)}{n^s} = \frac{1}{\zeta(s)}D_f(s)$$

De donde entonces $\mathbb{E}[f^k(N_s)] = \frac{1}{\zeta(s)}D_{f^k}(s)$ por lo que:

$$Var[f(N_s)] = \mathbb{E}[f^2(N_2)] + \mathbb{E}^2[f(N_2)] = \frac{1}{\zeta(s)}D_{f^2}(s) + \frac{1}{\zeta^2(s)}D_f^2(s)$$

Entender a las series de Dirichlet nos ayudara a estudiar esperanzas y varianzas

Series de Dirichlet

¿Y entonces porque funciones multiplicativas?

Teorema 2 (Producto de Euler)

Sea $f : \mathbb{Z}^+ \to \mathbb{C}$ multiplicativa, entonces:

$$D_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \sum_{m=0}^{\infty} \frac{f(p^m)}{p^{ms}}$$

y si f es completamente multiplicativa, entonces:

$$D_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 - \frac{f(p)}{p^s}\right)^{-1}$$

Demostración. –

Sabemos que $\mathbb{E}[f(N_s)] = \frac{1}{\zeta(s)} D_f(s)$ pero por otro lado, dado que f es multiplicativa

$$\Rightarrow \mathbb{E}[f(N_s)] = \mathbb{E}[\prod_p f(p^{\upsilon_p(N_s)})] = \prod_p \mathbb{E}[f(p^{\upsilon_p(N_s)})] = \prod_p \sum_{m=0}^{\infty} f(p^m) p^{-ms}(1 - p^{-s})$$

Series de Dirichlet

$$\Rightarrow \quad \mathbb{E}[f(N_s)] = \prod_p (1-p^{-s}) \sum_{m=0}^{\infty} f(p^m) p^{-ms} = \prod_p (1-p^{-s}) \prod_p \sum_{m=0}^{\infty} f(p^m) p^{-ms}$$

$$= \frac{1}{\zeta(s)} \prod_p \sum_{m=0}^{\infty} \frac{f(p^m)}{p^{ms}}$$

$$\therefore \quad \frac{1}{\zeta(s)} D_f(s) = \mathbb{E}[f(N_s)] = \frac{1}{\zeta(s)} \prod_p \sum_{m=0}^{\infty} \frac{f(p^m)}{p^{ms}} \qquad \Rightarrow \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \sum_{m=0}^{\infty} \frac{f(p^m)}{p^{ms}}$$

y si f es completamente multiplivativa, entonces

$$\Rightarrow \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \sum_{m=0}^{\infty} \frac{f(p^m)}{p^{ms}} = \prod_p \sum_{m=0}^{\infty} \frac{f(p)^m}{p^{ms}} = \prod_p \sum_{m=0}^{\infty} \left[\frac{f(p)}{p^s} \right]^m = \prod_p \left(1 - \frac{f(p)}{p^s} \right)^{-1}$$

Series de Dirichlet

Ejemplo:

$\mu(n)$ es multiplicativa, por lo que

$$D_{\mu}(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{s}} = \prod_{p} \sum_{m=0}^{\infty} \frac{\mu(p^{m})}{p^{ms}} = \prod_{p} \left(1 - \frac{1}{p^{s}}\right) = \frac{1}{\zeta(s)}$$

Series de Dirichlet

¿Y la convolucion que pinta aqui?

Teorema 3

Sean $f, g : \mathbb{Z}^+ \to \mathbb{C}$, entonces se tiene que:

$$D_{f*g}(s) = D_f(s)D_g(s)$$

$$\sum_{n=1}^{\infty} \frac{(f*g)(n)}{n^s} = \left(\sum_{n=1}^{\infty} \frac{f(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{g(n)}{n^s}\right)$$

Ejemplos:

- $$\sum_{n=1}^{\infty} \frac{\sigma_0(n)}{n^s} = \sum_{n=1}^{\infty} \frac{(1*1)(n)}{n^s} = \left(\sum_{n=1}^{\infty} \frac{1(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{1(n)}{n^s}\right) = \zeta^2(s)$$
- $$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{(\mu * \mathrm{id})(n)}{n^s} = \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{\mathrm{id}(n)}{n^s}\right) = \frac{\zeta(s-1)}{\zeta(s)}$$

Series de Dirichlet

Resumen:

Tenemos el problema de calcular

$$\mathbb{E}[f(N_s)] = \frac{1}{\zeta(s)} D_f(s) \qquad Var[f(N_s)] = \frac{1}{\zeta(s)} D_{f^2}(s) + \frac{1}{\zeta^2(s)} D_f^2(s)$$

y tenemos:

En general

$$D_{f*g}(s) = D_f(s) D_g(s)$$

Si f es multiplicativa

$$D_f(s) = \prod_p \sum_{m=0}^{\infty} \frac{f(p^m)}{p^{ms}}$$

Esperanzas y Varianzas

Entonces obtenemos de forma inmediata los siguientes:

- $\mathbb{E}[1(N_s)] = \dfrac{1}{\zeta(s)} D_1(s) = 1$
- $\mathbb{E}[\mathrm{id}(N_s)] = \mathbb{E}[N_s] = \dfrac{1}{\zeta(s)} D_{\mathrm{id}}(s) = \dfrac{\zeta(s-1)}{\zeta(s)}$
- $\mathbb{E}[\mathrm{id}_k(N_s)] = \mathbb{E}[N_s{}^k] = \dfrac{1}{\zeta(s)} D_{\mathrm{id}_k}(s) = \dfrac{\zeta(s-k)}{\zeta(s)}$
- $\mathbb{E}[\mu(N_s)] = \dfrac{1}{\zeta(s)} D_\mu(s) = \dfrac{1}{\zeta^2(s)}$
- $\mathbb{E}[\sigma_k(N_s)] = \dfrac{1}{\zeta(s)} D_{\sigma_k}(s) = \dfrac{1}{\zeta(s)} D_{\mathrm{id}_k * 1}(s) = \dfrac{1}{\zeta(s)} D_{\mathrm{id}_k}(s) D_1(s) = \dfrac{1}{\zeta(s)} \zeta(s-k)\zeta(s) = \zeta(s-k)$
- $\mathbb{E}[\varphi(N_s)] = \dfrac{1}{\zeta(s)} D_\varphi(s) = \dfrac{1}{\zeta(s)} D_{\mathrm{id}*\mu}(s) = \dfrac{1}{\zeta(s)} D_{\mathrm{id}}(s) D_\mu(s) = \dfrac{1}{\zeta(s)} \zeta(s-1) \dfrac{1}{\zeta(s)} = \dfrac{\zeta(s-1)}{\zeta^2(s)}$

Esperanzas y Varianzas

Definición 3

Se define a la funcion de Liouville como:

$$\lambda(n) = (-1)^{\Omega(n)}$$

$\otimes$ $\sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{si } n = k^2 \\ 0 & \text{e.o.c} \end{cases} \Rightarrow 1 * \lambda = \mathbb{I}_{n=k^2} \Rightarrow \lambda = \mathbb{I}_{n=k^2} * \mu$

- $\mathbb{E}[\lambda(N_s)] = \frac{1}{\zeta(s)} D_\lambda(s) = \frac{1}{\zeta(s)} D_{\mathbb{I}*\mu}(s) = \frac{1}{\zeta(s)} D_{\mathbb{I}}(s) D_\mu(s) = \frac{1}{\zeta^2(s)} \sum_{n=1}^{\infty} \frac{\mathbb{I}_{n=k^2}(n)}{n^s}$

$$= \frac{1}{\zeta^2(s)} \sum_{k=1}^{\infty} \frac{1}{(k^2)^s} = \frac{1}{\zeta^2(s)} \sum_{k=1}^{\infty} \frac{1}{k^{2s}} = \frac{\zeta(2s)}{\zeta^2(s)}$$

Esperanzas y Varianzas

Definición 4

Se define a la funcion de Von Mangoldt como:

$$\Lambda(n) = \begin{cases} \log p & \text{si } n = p^{\alpha} \\ 0 & \text{e.o.c} \end{cases}$$

$$\otimes \sum_{d|n} \Lambda(d) = \sum_{p|n} \Lambda(p^{\upsilon_p(n)}) = \sum_{p|n} \sum_{m=0}^{\upsilon_p(n)} \Lambda(p^m) = \sum_{p|n} \sum_{m=0}^{\upsilon_p(n)} \log p = \sum_{p|n} \upsilon_p(n) \log p$$

$$= \sum_{p|n} \log(p^{\upsilon_p(n)}) = \log\left(\prod_{p|n} p^{\upsilon_p(n)}\right) = \log n$$

$$\therefore \quad 1 * \Lambda = \log \quad \Leftrightarrow \quad \Lambda = \log * \mu$$

Esperanzas y Varianzas

- $\mathbb{E}[\Lambda(N_s)] = \dfrac{1}{\zeta(s)} D_\Lambda(s) = \dfrac{1}{\zeta(s)} D_{\log * \mu}(s) = \dfrac{1}{\zeta(s)} D_{\log}(s) D_\mu(s)$

Recordando que $\zeta(s) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} \Rightarrow \zeta'(s) = \sum_{n=1}^{\infty} \frac{-\log n}{n^s} \quad \therefore \sum_{n=1}^{\infty} \frac{\log n}{n^s} = -\zeta'(s)$

$$\Rightarrow \quad \mathbb{E}[\Lambda(N_s)] = \frac{1}{\zeta(s)}(-\zeta'(s))\frac{1}{\zeta(s)} = -\frac{\zeta'(s)}{\zeta^2(s)}$$

Esperanzas y Varianzas

¿Y la varianza?

$$Var[f(N_s)] = \frac{1}{\zeta(s)} D_{f^2}(s) + \frac{1}{\zeta^2(s)} D_f^2(s)$$ Depende de f....

Proposición 4

$$Var[\mu(N_s)] = \frac{1}{\zeta(2s)} + \frac{1}{\zeta^4(s)}$$

Demostración. –

Como μ es multiplicativa $\Rightarrow$ μ^2 es multiplicativa

$$\Rightarrow \quad D_{\mu^2}(s) = \prod_p \sum_{m=0}^{\infty} \frac{\mu^2(p^m)}{p^{ms}} = \prod_p \left(1 + \frac{1}{p^s}\right)$$ ¿$\prod_p \left(1 + \frac{1}{p^s}\right)$?

Sabemos que $\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) \Rightarrow \frac{1}{\zeta(2s)} = \prod_p \left(1 - \frac{1}{p^{2s}}\right) = \prod_p \left(1 - \frac{1}{p^s}\right) \prod_p \left(1 + \frac{1}{p^s}\right)$

Esperanzas y Varianzas

$$\Rightarrow \quad \frac{1}{\zeta(2s)} = \frac{1}{\zeta(s)} \prod_p \left(1 + \frac{1}{p^s}\right) \qquad \therefore \quad \frac{\zeta(s)}{\zeta(2s)} = \prod_p \left(1 + \frac{1}{p^s}\right) = D_{\mu^2}(s)$$

$$\therefore \quad Var[\mu(N_s)] = \frac{1}{\zeta(s)} D_{\mu^2}(s) + \frac{1}{\zeta^2(s)} D_{\mu}^2(s) = \frac{1}{\zeta(s)} \frac{\zeta(s)}{\zeta(2s)} + \frac{1}{\zeta^2(s)} \frac{1}{\zeta^2(s)} = \frac{1}{\zeta(2s)} + \frac{1}{\zeta^4(s)}$$

Obs. –

$$D_{\lambda}(s) = \frac{\zeta(2s)}{\zeta(s)} = \left(\frac{\zeta(s)}{\zeta(2s)}\right)^{-1} = \left(D_{\mu^2}(s)\right)^{-1} \qquad \Rightarrow \quad D_{\lambda}(s) D_{\mu^2}(s) = 1$$

$$\Rightarrow \quad D_{\lambda * \mu^2}(s) = 1$$

$$\therefore \quad \mu^2 = \lambda^{-1} \text{ en la convolucion}$$

Esperanzas y Varianzas

Proposición 5

$$Var[\sigma_0(N_s)] = \frac{\zeta^3(s)}{\zeta(2s)} + \zeta^2(s)$$

Todo se resumen en calcular $\mathbb{E}[{\sigma_0}^2(N_s)]$

$$\mathbb{E}[{\sigma_0}^2(N_s)] = \frac{1}{\zeta(s)} D_{{\sigma_0}^2}(s) = \frac{1}{\zeta(s)} \prod_p \sum_{m=0}^{\infty} \frac{\sigma_0^2(p^m)}{p^{ms}} = \frac{1}{\zeta(s)} \prod_p \sum_{m=0}^{\infty} \frac{(1+m)^2}{p^{ms}} \quad \ldots$$

Proposición 6

$$Var[\varphi(N_s)] = \frac{1}{\zeta(s)} \prod_p \frac{(p-1)^2}{p^2 - p^{4-s}} + \frac{\zeta^2(s-1)}{\zeta^4(s)}$$

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Esperanzas y Varianzas

Demostración. –

$$Var[\varphi(N_s)] = \frac{1}{\zeta(s)} D_{\varphi^2}(s) + \frac{1}{\zeta^2(s)} D_{\varphi}^2(s) = \frac{1}{\zeta(s)} D_{\varphi^2}(s) + \frac{1}{\zeta^2(s)} \left[\frac{\zeta(s-1)}{\zeta(s)} \right]^2$$

$$= \frac{1}{\zeta(s)} D_{\varphi^2}(s) + \frac{\zeta^2(s-1)}{\zeta^4(s)}$$

$$D_{\varphi^2}(s) = \prod_p \sum_{m=0}^{\infty} \frac{\varphi^2(p^m)}{p^{ms}} = \prod_p \sum_{m=0}^{\infty} \frac{(p^m - p^{m-1})^2}{p^{ms}} = \prod_p \sum_{m=0}^{\infty} \frac{p^{2m} - 2p^{2m-1} + p^{2m-2}}{p^{ms}}$$

$$= \prod_p \left[\frac{p^s}{p^s - p^2} - 2\frac{p^{s-1}}{p^s - p^2} + \frac{p^{s-2}}{p^s - p^2} \right] = \prod_p \frac{(p-1)^2}{p^2 - p^{4-s}}$$

$$Var[\varphi(N_s)] = \frac{1}{\zeta(s)} \prod_p \frac{(p-1)^2}{p^2 - p^{4-s}} + \frac{\zeta^2(s-1)}{\zeta^4(s)}$$