A Stochastic Approach to the Gamma Function

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Formula de Stirling

El objetivo de los proximos teoremas es demostrar la formula de Stirling

$$\boxed{\Gamma(t) \sim \sqrt{2\pi t}\, t^{t-1/2} e^{-t}}$$

Teorema 4

Sea $X_t \sim Gamma(t)$. Para todo $0 < t < \infty$ se tienen:

- $\psi'(t) = \operatorname{Var}\left[\log(X_t)\right] = \sum_{j=0}^{\infty} \frac{1}{(t+j)^2}$
- $\lim_{t\to\infty} \psi(t) - \log(t) = 0$

$$\left.\begin{matrix} \frac{1}{t} + \frac{1}{2t^2} + \frac{1}{6(t+1/14)^3} \\ \frac{1}{t} + \frac{1}{2t^2} + \frac{1}{6t^3} - \frac{1}{30t^5} \end{matrix}\right\} < \psi'(t) < \left\{\begin{matrix} \frac{1}{t} + \frac{1}{2t^2} + \frac{1}{6t^3} \\ \frac{1}{t} + \frac{1}{2t^2} + \frac{1}{6t^3} - \frac{1}{30(t+1/8)^5} \end{matrix}\right.$$

Formula de Stirling

Teorema 5

Para todo $t > 0$ se tiene:

$$\left.\begin{array}{r} \log(t) - \dfrac{1}{2t} - \dfrac{1}{12t^2} \\ \log(t) - \dfrac{1}{2t} - \dfrac{1}{12t^2} + \dfrac{1}{120(t+1/8)^4} \end{array}\right\} < \psi(t) < \left\{\begin{array}{l} \log(t) - \dfrac{1}{2t} - \dfrac{1}{12(t+1/4)^2} \\ \log(t) - \dfrac{1}{2t} - \dfrac{1}{12t^2} + \dfrac{1}{120t^4} \end{array}\right.$$

Dem. – Por el teorema 4, tenemos para $x > 0$

$$\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6(x+1/14)^3} < \psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} \quad \Rightarrow \quad \frac{1}{2x^2} + \frac{1}{6(x+1/14)^3} < \psi'(x) - \frac{1}{x} < \frac{1}{2x^2} + \frac{1}{6x^3}$$

por lo que para $0 < t < x$

$$\Rightarrow \quad \int_t^\infty \frac{1}{2x^2} + \frac{1}{6(x+1/14)^3}\,dx < \int_t^\infty \psi'(x) - \frac{1}{x}\,dx < \int_t^\infty \frac{1}{2x^2} + \frac{1}{6x^3}\,dx$$

Formula de Stirling

$$\Rightarrow \quad \left[-\frac{1}{2x} - \frac{1}{12(x+1/14)^2}\right]_t^\infty < \left[\psi(x) - \log(x)\right]_t^\infty < \left[-\frac{1}{2x} - \frac{1}{12x^2}\right]_t^\infty$$

$$\underset{\text{teo } 4}{\Rightarrow} \quad \frac{1}{2t} + \frac{1}{12(t+1/14)^2} < 0 - \left[\psi(t) - \log(t)\right] < \frac{1}{2t} + \frac{1}{12t^2}$$

$$\therefore \qquad \boxed{\log(t) - \frac{1}{2t} - \frac{1}{12t^2} < \psi(t) < \log(t) - \frac{1}{2t} - \frac{1}{12(t+1/14)^2}}$$

la segunda desigualdad se obtiene de la misma forma ■

A partir de esta ultima podemos obtener la formula de stirling

Formula de Stirling

Corolario (2da parte teorema 5)

Para todo $t > 0$ se tiene:

$$\left.\begin{array}{c}\dfrac{1}{12t+6/7}\\[2ex] \dfrac{1}{12t}-\dfrac{1}{360t^3}\end{array}\right\} < \log\left(\frac{\Gamma(t)}{\sqrt{2\pi}t^{t-1/2}e^{-t}}\right) < \left\{\begin{array}{c}\dfrac{1}{12t}\\[2ex] \dfrac{1}{12t}-\dfrac{1}{360(t+1/8)^3}\end{array}\right.$$

y por tanto

$$\Gamma(t) \sim \sqrt{2\pi}t^{t-1/2}e^{-t}$$

Dem. – Definimos $L(y) = \int_t^y \psi(x) - \log(x) + \frac{1}{2x}\,dx$ y veremos que $\lim_{y\to\infty} L(y)$ existe.

⊙ Notemos que $L(y)$ es decreciente

En efecto, por el Teorema 5:

$$\log(x) - \frac{1}{2x} - \frac{1}{12x^2} < \psi(x) < \log(x) - \frac{1}{2x} - \frac{1}{12(x+1/14)^2}$$

$$\Rightarrow \psi(x) - \log(x) + \frac{1}{2x} < -\frac{1}{12(x+1/14)^2} < 0$$

Formula de Stirling

por lo que $L'(y) = \psi(y) - \log(y) + \dfrac{1}{2y} < 0$

⊙ Notemos que $L(y)$ es acotada inferiormente

$$-\frac{1}{12x^2} < \psi(x) - \log(x) + \frac{1}{2x} < -\frac{1}{12(x+1/14)^2} \underset{(*)}{\Rightarrow} \frac{1}{12y} - \frac{1}{12t} < L(y) < \frac{1}{12(y+1/14)} - \frac{1}{12(t+1/14)}$$

$$\Rightarrow \quad -\frac{1}{12t} < L(y) \qquad \therefore \quad \lim_{y\to\infty} L(y) \text{ existe}$$

Ahora, por lo anterior $\underset{(*)}{\Rightarrow} \dfrac{1}{12y} - \dfrac{1}{12t} < L(y) < \dfrac{1}{12(y+1/14)} - \dfrac{1}{12(t+1/14)}$

$$\underset{y\to\infty}{\Rightarrow} -\frac{1}{12t} < \lim_{y\to\infty} L(y) < -\frac{1}{12(t+1/14)}$$

$$\Rightarrow \quad -\frac{1}{12t} < \int_t^{\infty} \psi(x) - \log(x) + \frac{1}{2x}\,dx < -\frac{1}{12(t+1/14)}$$

$$\Rightarrow \quad -\frac{1}{12t} < \left[\log\Gamma(x) - x\log(x) + x + \frac{1}{2}\log(x)\right]_t^{\infty} < -\frac{1}{12(t+1/14)}$$

Formula de Stirling

$$\Rightarrow \quad -\frac{1}{12t} < c - \left[\log\Gamma(t) - t\log(t) + t + \frac{1}{2}\log(t)\right] < -\frac{1}{12(t+1/14)}$$

pero $\log\Gamma(t) - t\log(t) + t + \dfrac{1}{2}\log(t) = \log\Gamma(t) - \log(t^t) + \log(e^t) + \log(t^{1/2})$

$$= \log\left(\frac{\Gamma(t)e^t t^{1/2}}{t^t}\right) = \log\left(\frac{\Gamma(t)}{t^{t-1/2}e^{-t}}\right)$$

$$\Rightarrow \quad -\frac{1}{12t} < c - \log\left(\frac{\Gamma(t)}{t^{t-1/2}e^{-t}}\right) < -\frac{1}{12(t+1/14)} \quad \Rightarrow \quad \frac{1}{12(t+1/14)} < \log\left(\frac{\Gamma(t)}{t^{t-1/2}e^{-t}}\right) - c < \frac{1}{12t}$$

¿Quien es c? de lo anterior tenemos que

$$c + \frac{1}{12(t+1/14)} < \log\left(\frac{\Gamma(t)}{t^{t-1/2}e^{-t}}\right) < c + \frac{1}{12t} \quad \underset{t\to\infty}{\Rightarrow} \quad c = \lim_{t\to\infty} \log\left(\frac{\Gamma(t)}{t^{t-1/2}e^{-t}}\right)$$

$$\Rightarrow \quad e^c = \lim_{t\to\infty} \frac{\Gamma(t)}{t^{t-1/2}e^{-t}} \underset{t\to 2t}{=} \lim_{t\to\infty} \frac{\Gamma(2t)}{(2t)^{2t-1/2}e^{-2t}}$$ usando la formula de duplicacion:

Formula de Stirling

$$\Gamma(2t) = 2^{2t-1}\frac{\Gamma(t)\Gamma(t+1/2)}{\Gamma(1/2)}$$

$$\Rightarrow \quad e^{c} = \lim_{t\to\infty}\frac{\Gamma(2t)}{(2t)^{2t-1/2}e^{-2t}} = \lim_{t\to\infty}\frac{1}{(2t)^{2t-1/2}e^{-2t}}2^{2t-1}\frac{\Gamma(t)\Gamma(t+1/2)}{\Gamma(1/2)}$$

$$= \lim_{t\to\infty}\frac{1}{\Gamma(1/2)}\Gamma(t)\Gamma(t+1/2)\frac{1}{t^{2t-1/2}e^{-2t}}\frac{2^{2t-1}}{2^{2t-1/2}}$$

$$= \lim_{t\to\infty}\frac{1}{\Gamma(1/2)}\frac{\Gamma(t)t^{t-1/2}e^{-t}}{t^{t-1/2}e^{-t}}\frac{\Gamma(t+1/2)(t+1/2)^{t+1/2-1/2}e^{-t-1/2}}{(t+1/2)^{t+1/2-1/2}e^{-t-1/2}}\frac{1}{t^{2t-1/2}e^{-2t}}\frac{2^{1/2}}{2}$$

$$= \frac{1}{\sqrt{\pi}}\frac{2^{1/2}}{2}e^{c}e^{c}\lim_{t\to\infty}\frac{(t^{t-1/2}e^{-t})(t+1/2)^{t}e^{-t-1/2}}{t^{2t-1/2}e^{-2t}}$$

$$= \frac{e^{2c}}{\sqrt{2\pi}}\lim_{t\to\infty}\frac{(t+1/2)^{t}e^{-2t-1/2}}{t^{t}e^{-2t}} = \frac{e^{2c}}{\sqrt{2\pi}}\lim_{t\to\infty}\frac{(t+1/2)^{t}e^{-1/2}}{t^{t}}$$

$$= \frac{e^{2c}}{\sqrt{2\pi}}e^{-1/2}\lim_{t\to\infty}\left(1+\frac{1/2}{t}\right)^{t} = \frac{e^{2c}}{\sqrt{2\pi}} \qquad \therefore \quad e^{c} = \frac{e^{2c}}{\sqrt{2\pi}} \quad \Leftrightarrow \quad c = \log(\sqrt{2\pi})$$

Regresando a la desigualdad...

Formula de Stirling

$$\frac{1}{12(t+1/14)} < \log\left(\frac{\Gamma(t)}{t^{t-1/2}e^{-t}}\right) - c < \frac{1}{12t} \quad \Rightarrow \quad \frac{1}{12(t+1/14)} < \log\left(\frac{\Gamma(t)}{\sqrt{2\pi}t^{t-1/2}e^{-t}}\right) < \frac{1}{12t}$$

finalmente tomando $t \to \infty$ $\quad \Rightarrow \quad \lim_{t\to\infty} \log\left(\frac{\Gamma(t)}{\sqrt{2\pi}t^{t-1/2}e^{-t}}\right) = 0$

$$\Leftrightarrow \quad \lim_{t\to\infty} \frac{\Gamma(t)}{\sqrt{2\pi}t^{t-1/2}e^{-t}} = 1 \quad \Leftrightarrow \quad \Gamma(t) \sim \sqrt{2\pi}t^{t-1/2}e^{-t}$$

$\blacksquare$

Formula producto de Gauss

Ahora probaremos la formula producto de Gauss, que generaliza a la formula de duplicacion

Teorema 6

Sea $p \geq 2$ un entero y $t > 0$. Sean $X_t, X_{t+1/p}, \dots, X_{t+(p-1)/p}$ variables gamma independientes. Entonces:

$$p^p \prod_{i=0}^{p-1} X_{t+i/p} = X_{pt}^p$$

$$p^{pt-1/2} \prod_{i=0}^{p-1} \Gamma(t + i/p) = (2\pi)^{(p-1)/2} \Gamma(pt)$$

Dem. –

Por el Teorema 2

$$p \log(X_{pt}) = -p\gamma + p \sum_{j=0}^{\infty} \left(\frac{1}{j+1} - \frac{Y_j}{j+pt} \right)$$

Separaremos los sumandos en congruencias

Formula producto de Gauss

para $j \equiv i(\mathrm{mod}\ p)$ con $i = 0,1,\ldots,p-1 \quad \Leftrightarrow \quad j = pk + i$ con $i = 0,1,\ldots,p-1$ y $k \geq 0$

$$\Rightarrow \quad p\log(X_{pt}) = -p\gamma + p\sum_{i=0}^{p-1}\sum_{k=0}^{\infty}\left(\frac{1}{pk+i+1} - \frac{Y_{pk+i}}{pk+i+pt}\right)$$

$$= -p\gamma + \sum_{i=0}^{p-1}\sum_{k=0}^{\infty}\left(\frac{p}{pk+i+1} - \frac{Y_{pk+i}}{k+(t+i/p)}\right)$$

$$= -p\gamma + \sum_{i=0}^{p-1}\sum_{k=0}^{\infty}\left(\frac{p}{pk+i+1} - \frac{1}{k+1} + \frac{1}{k+1} - \frac{Y_{pk+i}}{k+(t+i/p)}\right)$$

$$= \left[\sum_{i=0}^{p-1} -\gamma + \sum_{k=0}^{\infty}\left(\frac{1}{k+1} - \frac{Y_{pk+i}}{k+(t+i/p)}\right)\right] + \sum_{i=0}^{p-1}\sum_{k=0}^{\infty}\left(\frac{p}{pk+i+1} - \frac{1}{k+1}\right)$$

$$= \sum_{i=0}^{p-1}\log(X_{t+i/p}) + \sum_{i=0}^{p-1}\sum_{k=0}^{\infty}\left(\frac{p}{pk+i+1} - \frac{1}{k+1}\right) = \sum_{i=0}^{p-1}\log(X_{t+i/p}) + S$$

¿Que pasa con el segundo sumando?

Formula producto de Gauss

recordemos que por el Teorema 4

$$\psi'(x) = \sum_{k=0}^{\infty} \frac{1}{(x+k)^2}$$

dicha serie converge unfiormemente para $x > 0$, de donde

$$\int_1^t \psi'(x)dx = \int_1^t \sum_{k=0}^{\infty} \frac{1}{(x+k)^2} dx$$

$$\Rightarrow \quad \psi(t) - \psi(1) = \int_1^t \psi'(x)dx \; = \int_1^t \sum_{k=0}^{\infty} \frac{1}{(x+k)^2} dx \; = \sum_{k=0}^{\infty} \int_1^t \frac{1}{(x+k)^2}\, dx \; = \sum_{k=0}^{\infty} \left[-\frac{1}{x+k} \right]_1^t$$

$$= \sum_{k=0}^{\infty} \left[-\frac{1}{t+k} + \frac{1}{1+k} \right] = -\sum_{k=0}^{\infty} \left[\frac{1}{t+k} - \frac{1}{k+1} \right]$$

$$\therefore \quad \psi(t) = \psi(1) - \sum_{k=0}^{\infty} \left[\frac{1}{k+t} - \frac{1}{k+1} \right]$$

Formula producto de Gauss

con esto obtenemos $\quad \therefore \sum_{k=0}^{\infty}\left[\frac{1}{k+t}-\frac{1}{k+1}\right]=\psi(1)-\psi(t)$

$$S=\sum_{i=0}^{p-1}\sum_{k=0}^{\infty}\left(\frac{p}{pk+i+1}-\frac{1}{k+1}\right)=\sum_{i=0}^{p-1}\sum_{k=0}^{\infty}\left(\frac{1}{k+(i+1)/p}-\frac{1}{k+1}\right)$$

$$=\sum_{i=0}^{p-1}\Big(\psi(1)-\psi((i+1)/p)\Big)=p\psi(1)-\sum_{i=0}^{p-1}\psi(\frac{i+1}{p})$$

que es una constante finita.

Con esto obtenemos

$$\Rightarrow \quad p\log(X_{pt})=\sum_{i=0}^{p-1}\log(X_{t+i/p})+S \quad \Rightarrow \quad \log(X_{pt}^{p})=\log\left(\prod_{i=0}^{p-1}X_{t+i/p}\right)+S$$

$$\therefore \quad X_{pt}^{p}=e^{S}\prod_{i=0}^{p-1}X_{t+i/p}$$

Formula producto de Gauss

$$E\left[X_t^r\right] = \frac{\Gamma(t+r)}{\Gamma(t)}$$

y para saber quien es e^S, sacamos esperanzas

$$\Rightarrow \mathbb{E}\left[X_{pt}^p\right] = e^S \mathbb{E}\left[\prod_{i=0}^{p-1} X_{t+i/p}\right] = e^S \prod_{i=0}^{p-1} \mathbb{E}\left[X_{t+i/p}\right]$$

$$\Rightarrow \frac{\Gamma(pt+p)}{\Gamma(pt)} = e^S \prod_{i=0}^{p-1} \frac{\Gamma(t+i/p+1)}{\Gamma(t+i/p)} = e^S \prod_{i=0}^{p-1} \frac{(t+i/p)\Gamma(t+i/p)}{\Gamma(t+i/p)} = e^S \prod_{i=0}^{p-1} (t+i/p)$$

$$= e^S \prod_{i=0}^{p-1} \frac{1}{p}(pt+i) = e^S \frac{1}{p^p} \prod_{i=0}^{p-1} (pt+i)$$

pero del lado izquierdo

$$\frac{\Gamma(pt+p)}{\Gamma(pt)} = \frac{(pt+p-1)\Gamma(pt+p-1)}{\Gamma(pt)} = \frac{(pt+p-1)(pt+p-2)\Gamma(pt+p-2)}{\Gamma(pt)}$$

$$= \frac{(pt+p-1)(pt+p-2)\ldots(pt)\Gamma(pt)}{\Gamma(pt)} = \prod_{i=0}^{p-1}(pt+i) \qquad \therefore \quad e^S = p^p$$

de donde $\boxed{p^p \prod_{i=0}^{p-1} X_{t+i/p} = X_{pt}^p}$

Formula producto de Gauss

Finalmente, de lo anterior sea $t = \dfrac{1}{p}$ $\Rightarrow$ $p^p \prod_{i=1}^{p} X_{i/p} = X_1^p$

elevando a r y tomando esperanza

$$\Rightarrow \; p^{pr} \prod_{i=i}^{p} X_{i/p}^{r} = X_1^{pr} \quad \Rightarrow \; \mathbb{E}\left[p^{pr} \prod_{i=1}^{p} X_{i/p}^{r} \right] = \mathbb{E}\left[X_1^{pr} \right] \quad \Rightarrow \; p^{pr} \prod_{i=1}^{p} \frac{\Gamma(i/p + r)}{\Gamma(i/p)} = \frac{\Gamma(1 + pr)}{\Gamma(1)}$$

$$\Rightarrow \; p^{pr} \prod_{i=1}^{p} \Gamma(i/p + r) = \Gamma(pr + 1) \prod_{i=1}^{p} \Gamma(i/p)$$

Obs. – Notemos que tenemos una constante involucrada $\prod_{i=1}^{p} \Gamma(i/p) = \dfrac{p^{pr}}{\Gamma(pr+1)} \prod_{i=1}^{p} \Gamma(i/p + r)$

que no depende de r, por lo que podemos tomar $r \to \infty$

$$\prod_{i=1}^{p} \Gamma(i/p) = (2\pi)^{(p-1)/2} p^{-1/2}$$

Formula producto de Gauss

de donde

$$\Rightarrow \quad p^{pr}\prod_{i=1}^{p}\Gamma(i/p+r) = \Gamma(pr+1)(2\pi)^{(p-1)/2}p^{-1/2}$$

$$\Rightarrow \quad p^{pr}\frac{\Gamma(r+1)}{\Gamma(r)}\prod_{i=0}^{p-1}\Gamma(i/p+r) = pr\Gamma(pr)(2\pi)^{(p-1)/2}p^{-1/2}$$

$$\Rightarrow \quad \frac{p^{pr}}{pr}p^{1/2}\frac{\Gamma(r+1)}{\Gamma(r)}\prod_{i=0}^{p-1}\Gamma(i/p+r) = (2\pi)^{(p-1)/2}\Gamma(pr)$$

$$\Rightarrow \quad p^{pr-1/2}\prod_{i=0}^{p-1}\Gamma(i/p+r) = (2\pi)^{(p-1)/2}\Gamma(pr) \qquad \blacksquare$$

Formula de reflexión

Finalmente se concluye con una prueba de la formula de reflexion, para esta necesitaremos la expresion en producto infinito de la funcion gamma

Teorema 7

Para $t > -1$, tenemos

$$\mathbb{E}[e^{t\log(Y)}] = \Gamma(1+t) = e^{-\gamma t}\prod_{j=1}^{\infty}\frac{1}{1+t/j}e^{t/j}$$

donde $Y \sim \exp(1)$

Dem. – Tenemos

$$\mathbb{E}[e^{t\log(Y)}] = \int_0^\infty e^{t\log(y)}e^{-y}dy = \int_0^\infty y^t e^{-y}dy = \Gamma(t+1)$$

Ahora, recordando que por el Teorema 2, dada $X_t \sim Gamma(t)$

$$\log(X_t) = -\gamma + \left[\sum_{j=0}^{n-1}\frac{1}{j+1} - \frac{Y_j}{j+t}\right] + \log(X_{t+n}/(t+n)) + d_{n,t}\,, \quad \text{con } \left|d_{n,t}\right| < \frac{t+1}{n}$$

Formula de reflexión

y como $Y \sim \exp(1) \sim \text{gamma}(1)$

$$\log(Y) = \log(X_1) = -\gamma + \left[\sum_{j=0}^{n-1} \frac{1}{j+1} - \frac{Y_j}{j+1}\right] + \log(X_{n+1}/(1+n)) + d_{n,1}$$

$$\Rightarrow \Gamma(1+t) = \mathbb{E}[e^{t\log(Y)}] = \mathbb{E}\left[e^{-\gamma t + t\left[\sum_{j=0}^{n-1} \frac{1}{j+1} - \frac{Y_j}{j+1}\right] + t\log(X_{n+1}/(1+n)) + td_{n,1}}\right]$$

$$= \mathbb{E}\left[e^{-\gamma t} e^{t\left[\sum_{j=0}^{n-1} \frac{1}{j+1} - \frac{Y_j}{j+1}\right]} e^{t\log(X_{n+1}/(1+n))} e^{td_{n,1}}\right]$$

$$= e^{-\gamma t}\mathbb{E}\left[\left(\frac{X_{n+1}}{1+n}\right)^t \prod_{j=0}^{n-1} e^{\frac{t}{j+1} - \frac{tY_j}{j+1}}\right] e^{td_{n,1}} = e^{-\gamma t}\left(\prod_{j=0}^{n-1} \mathbb{E}\left[e^{\frac{t}{j+1} - \frac{tY_j}{j+1}}\right]\right) \mathbb{E}\left[\left(\frac{X_{n+1}}{1+n}\right)^t\right] e^{td_{n,1}}$$

$$= e^{-\gamma t}\left(\prod_{j=0}^{n-1} e^{\frac{t}{j+1}} \prod_{j=0}^{n-1} \mathbb{E}\left[e^{-\frac{t}{j+1}Y_j}\right]\right) \mathbb{E}\left[\left(\frac{X_{n+1}}{1+n}\right)^t\right] e^{td_{n,1}}$$

$$= e^{-\gamma t}\left(\prod_{j=0}^{n-1} e^{\frac{t}{j+1}} \prod_{j=0}^{n-1} \frac{1}{1+\left[\frac{t}{j+1}\right]}\right) \mathbb{E}\left[\left(\frac{X_{n+1}}{1+n}\right)^t\right] e^{td_{n,1}}$$

Formula de reflexión

$$= e^{-\gamma t}\left(\prod_{j=0}^{n-1}\frac{1}{1+t/(j+1)}e^{t/(j+1)}\right)\mathbb{E}\left[\left(\frac{X_{n+1}}{n+1}\right)^t\right]e^{td_{n,1}}$$

como $\left|d_{n,1}\right| < \frac{2}{n} \Rightarrow \lim_{n\to\infty} td_{n,1} = 0 \Rightarrow \lim_{n\to\infty} e^{td_{n,1}} = 1$ y usando la formula de Stirling

$$\lim_{n\to\infty}\mathbb{E}\left[\left(\frac{X_{n+1}}{n+1}\right)^t\right] = \lim_{n\to\infty}\frac{1}{(n+1)^t}\mathbb{E}\left[X_{n+1}^t\right] = \lim_{n\to\infty}\frac{1}{n^t}\frac{\Gamma(n+1+t)}{\Gamma(n+1)} = 1$$ por lo que

$$\mathbb{E}[e^{t\log(Y)}] = \lim_{n\to\infty}\mathbb{E}[e^{t\log(Y)}] = \lim_{n\to\infty} e^{-\gamma t}\left(\prod_{j=0}^{n-1}\frac{1}{1+t/(j+1)}e^{t/(j+1)}\right)\mathbb{E}\left[\left(\frac{Y}{1+n}\right)^t\right]e^{td_{n,1}}$$

$$= e^{-\gamma t}\left(\prod_{j=0}^{\infty}\frac{1}{1+t/(j+1)}e^{t/(j+1)}\right) = e^{-\gamma t}\left(\prod_{j=1}^{\infty}\frac{1}{1+t/j}e^{t/j}\right)$$

$\blacksquare$

Formula de reflexión

Finalmente

Teorema 8

Sean $Y_1, Y_2 \sim \exp(1)$ independientes. Para $0 < t < 1$ se tiene

$$\mathbb{E}\left[e^{t\{\log(Y_1)-\log(Y_2)\}}\right] = \Gamma(1+t)\Gamma(1-t) = \prod_{j=1}^{\infty}\frac{1}{1-t^2/j^2} = \frac{\pi t}{\sin(\pi t)}$$

Dem. – De forma directa

$$\mathbb{E}\left[e^{t\{\log(Y_1)-\log(Y_2)\}}\right] = \mathbb{E}\left[e^{t\log(Y_1)}\right]\mathbb{E}\left[e^{-t\log(Y_2)}\right] = \Gamma(1+t)\Gamma(1-t)$$

y por el teorema 7

$$\Gamma(1+t)\Gamma(1-t) = e^{-\gamma t}\left(\prod_{j=1}^{\infty}\frac{1}{1+t/j}e^{t/j}\right)e^{-\gamma(-t)}\left(\prod_{j=1}^{\infty}\frac{1}{1+(-t)/j}e^{(-t)/j}\right)$$

$$= \left(\prod_{j=1}^{\infty}\frac{1}{1+t/j}\right)\left(\prod_{j=1}^{\infty}\frac{1}{1+(-t)/j}\right) = \prod_{j=1}^{\infty}\frac{1}{1-t^2/j^2}$$

$\blacksquare$

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