EULER'S FORMULAE FOR $\zeta(2n)$ AND PRODUCTS OF CAUCHY VARIABLES

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- Un vistazo rápido a la prueba de Euler

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Números de Bernoulli

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\vdots$$

$$1^m + 2^m + 3^m + \cdots + n^m = ¿? \quad \text{con } m \in \mathbb{Z}^+$$

LLamemos $S_m(n) = 1^m + 2^m + \cdots + (n-1)^m, \quad n \geq 1, \; m \geq 0$

$$\Rightarrow \quad S_{m+1}(n+1) = 1 + \sum_{i=0}^{m+1} \binom{m+1}{i} S_i(n)$$

$$\Rightarrow \quad S_m(n) = \frac{1}{m+1}\left[n^{m+1} - 1 - \sum_{i=0}^{m-1} \binom{m+1}{i} S_i(n)\right]$$

Números de Bernoulli

$$S_1(n) = \frac{1}{2}n^2 - \frac{1}{2}n$$

$$S_2(n) = \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$$

$$S_3(n) = \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2$$

$$S_4(n) = \frac{1}{5}n^5 - \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

$$S_5(n) = \frac{1}{6}n^6 - \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$$

Números de Bernoulli

- $S_1(n) = \frac{1}{2}\left[\binom{2}{0}n^2 + \binom{2}{1}(-\frac{1}{2})n\right]$
- $S_2(n) = \frac{1}{3}\left[\binom{3}{0}n^3 + \binom{3}{1}(-\frac{1}{2})n^2 + \binom{3}{2}(\frac{1}{6})n\right]$
- $S_3(n) = \frac{1}{4}\left[\binom{4}{0}n^4 + \binom{4}{1}(-\frac{1}{2})n^3 + \binom{4}{2}(\frac{1}{6})n^2 + \binom{4}{3}\cdot 0 \cdot n\right]$
- $S_4(n) = \frac{1}{5}\left[\binom{5}{0}n^5 + \binom{5}{1}(-\frac{1}{2})n^4 + \binom{5}{2}(\frac{1}{6})n^3 + \binom{5}{3}\cdot 0 \cdot n^2 + \binom{5}{4}(-\frac{1}{30})n\right]$
- $S_5(n) = \frac{1}{6}\left[\binom{6}{0}n^6 + \binom{6}{1}(-\frac{1}{2})n^5 + \binom{6}{2}(\frac{1}{6})n^4 + \binom{6}{3}\cdot 0 \cdot n^3 + \binom{6}{4}(-\frac{1}{30})n^2 + \binom{6}{5}\cdot 0 \cdot n\right]$

Números de Bernoulli

Pero $\quad S_m(n) = \dfrac{1}{m+1}\left[n^{m+1} - 1 - \sum_{i=0}^{m-1} \binom{m+1}{i} S_i(n) \right]$

$$\Rightarrow \quad S'_m(0) = -\frac{1}{m+1} \sum_{i=0}^{m-1} \binom{m+1}{i} S_i{}'(0)$$

Definición

Se definen a los Numeros de Bernoulli como:

$$B_0 = 1, \qquad B_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k$$

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = -\frac{1}{30},\ B_5 = 0$$

Números de Bernoulli

Proposición 1

Los numeros de Bernoulli cumplen:

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = \frac{z}{e^z - 1}, \quad \forall |z| < \pi$$

Proposición 2

Se tiene que $B_{2n+1} = 0 \quad \forall n \geq 1$

Demostración. –

Consideremos a la función $\coth(z) := \dfrac{e^z + e^{-z}}{e^z - e^{-z}}$

$$\Rightarrow \quad \coth(z) - 1 = \frac{2e^{-z}}{e^z - e^{-z}} = \frac{2}{e^{2z} - 1} \quad \Rightarrow \quad \frac{z}{2}\coth\left(\frac{z}{2}\right) = \frac{z}{2} + \frac{z}{e^z - 1}$$

Números de Bernoulli

$$= \quad \cdots \quad z^n$$

pero el lado izquierdo ...

Teorema 1

... Atque si porrò ad altiores gradatim potestates pergere, levique negotio sequentem adornare laterculum licet :

Summae Potestatum

$\int n = \frac{1}{2}nn + \frac{1}{2}n$
$\int nn = \frac{1}{3}n^3 + \frac{1}{2}nn + \frac{1}{6}n$
$\int n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}nn$
$\int n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$
$\int n^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}nn$
$\int n^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$
$\int n^7 = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}nn$
$\int n^8 = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n$
$\int n^9 = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{1}{12}nn$
$\int n^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - 1n^7 + 1n^5 - \frac{1}{2}n^3 + \frac{5}{66}n$

Quin imò qui legem progressionis inibi attentuis ensperexit, eundem etiam continuare poterit absque his ratiociniorum ambabimus : Sumtâ enim c pro potestatis cujuslibet exponente, fit summa omnium n^c seu

$$\int n^c = \frac{1}{c+1}n^{c+1} + \frac{1}{2}n^c + \frac{c}{2}An^{c-1} + \frac{c\cdot c-1\cdot c-2}{2\cdot 3\cdot 4}Bn^{c-3}$$
$$+\frac{c\cdot c-1\cdot c-2\cdot c-3\cdot c-4}{2\cdot 3\cdot 4\cdot 5\cdot 6}Cn^{c-5}$$
$$+\frac{c\cdot c-1\cdot c-2\cdot c-3\cdot c-4\cdot c-5\cdot c-6}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}Dn^{c-7} \ldots \text{\& ita deinceps,}$$

exponentem potestatis ipsius n continué minuendo binario, quosque perveniatur ad n vel nn. Literae capitales A, B, C, D & c. ordine denotant coëfficientes ultimorum terminorum pro $\int nn$, $\int n^4$, $\int n^6$, $\int n^8$, & c. nempe

$$A = \frac{1}{6}, B = -\frac{1}{30}, C = \frac{1}{42}, D = -\frac{1}{30}.$$

Prueba de Euler

Consideremos

$$\sin(z) = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\pi^2 n^2}\right)$$

$$\Rightarrow \quad -i\sin(iz) = -i(iz)\prod_{n=1}^{\infty}\left(1 - \frac{(iz)^2}{\pi^2 n^2}\right) = z\prod_{n=1}^{\infty}\left(1 + \frac{z^2}{\pi^2 n^2}\right)$$

$$\sinh(z)$$

$$\Rightarrow \quad \frac{\sinh(z)}{z} = \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{\pi^2 n^2}\right)$$

$$\Rightarrow \quad \log\left(\frac{\sinh(z)}{z}\right) = \log\prod_{n=1}^{\infty}\left(1 + \frac{z^2}{\pi^2 n^2}\right) = \sum_{n=1}^{\infty}\log\left(1 + \frac{z^2}{\pi^2 n^2}\right) + C$$

Prueba de Euler

Derivando de ambos lados obtenemos que

$$\Rightarrow \quad \coth(z) - \frac{1}{z} = 2\sum_{n=1}^{\infty} \frac{z}{z^2 + \pi^2 n^2} = 2\sum_{n=1}^{\infty} \frac{z}{\pi^2 n^2} \frac{1}{(z/\pi n)^2 + 1}$$

$$= 2\sum_{n=1}^{\infty} \frac{z}{\pi^2 n^2} \sum_{m=1}^{\infty} (-1)^{m-1} (\frac{z}{\pi n})^{2m-2}$$

$$\Rightarrow \quad \coth(z) - \frac{1}{z} = 2\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{m-1} \frac{z^{2m-1}}{\pi^{2m} n^{2m}} = 2\sum_{m=1}^{\infty} (-1)^{m-1} \frac{z^{2m-1}}{\pi^{2m}} \sum_{n=1}^{\infty} \frac{1}{n^{2m}}$$

$$= \sum_{m=1}^{\infty} (-1)^{m-1} \frac{2z^{2m-1}}{\pi^{2m}} \zeta(2m)$$

Prueba de Euler

$$\therefore \quad \coth(z) = \frac{1}{z} + \sum_{n=1}^{\infty} 2(-1)^{n-1} \frac{z^{2n-1}}{\pi^{2n}} \zeta(2n)$$

pero por el Teo 1

$$\coth(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{B_{2n} 2^{2n}}{(2n)!} z^{2n-1}, \quad \forall |z| < \pi$$

$$\Rightarrow \quad 2(-1)^{n-1} \frac{1}{\pi^{2n}} \zeta(2n) = \frac{B_{2n} 2^{2n}}{(2n)!} \qquad \Rightarrow \quad \zeta(2n) = \frac{(-1)^{n-1} 2^{2n-1} B_{2n}}{(2n)!} \pi^{2n} \quad \forall n \geq 1$$

Variables logarítmicas

Consideremos dos variables de Cauchy $\mathbb{C}_1$ y $\mathbb{C}_2$ independientes y sea:

$$\Lambda := \log\left|\mathbb{C}_1\mathbb{C}_2\right|$$

Sabemos que la funcion de densidad de $\mathbb{C}_1\mathbb{C}_2$, viene dada por:

$$\psi(x) = \frac{2\log|x|}{\pi^2(x^2-1)}$$

Entonces la esperanza de $(\Lambda)^{2n}$ vendra dada por

$$\mathbb{E}[(\Lambda)^{2n}] = \int_{-\infty}^{\infty}\left(\log\left|x\right|\right)^{2n}\psi(x)dx \quad = \int_{-\infty}^{\infty}\left(\log\left|x\right|\right)^{2n}\frac{2\log|x|}{\pi^2(x^2-1)}dx$$

$$= 2\int_0^{\infty}\frac{2\log^{2n+1}(x)}{\pi^2(x^2-1)}dx = \frac{4}{\pi^2}\int_0^{\infty}\frac{\log^{2n+1}(x)}{x^2-1}dx$$

Variables logarítmicas

$$= \frac{4}{\pi^2}\int_0^1 \frac{\log^{2n+1}(x)}{x^2-1}dx + \frac{4}{\pi^2}\int_1^\infty \frac{\log^{2n+1}(x)}{x^2-1}dx$$

Tomando $x = \frac{1}{y}$ en la primer integral $\Rightarrow \quad dx = -\frac{1}{y^2}dy$

$$\int_0^1 \frac{\log^{2n+1}(x)}{x^2-1}dx = \int_\infty^1 \frac{\log^{2n+1}(1/y)}{(1/y)^2-1}\left(-\frac{1}{y^2}\right)dx \;= \int_1^\infty \frac{\log^{2n+1}(y)}{y^2-1}dx$$

$$\Rightarrow \quad \mathbb{E}[(\Lambda)^{2n}] = \frac{8}{\pi^2}\int_1^\infty \frac{\log^{2n+1}(x)}{x^2-1}dx$$

Tomando $x = e^{-u} \quad \Rightarrow \quad dx = -e^{-u}du$

$$\Rightarrow \quad \mathbb{E}[(\Lambda)^{2n}] = \frac{8}{\pi^2}\int_0^{-\infty} \frac{\log^{2n+1}(e^{-u})}{(e^{-u})^2-1}(-e^{-u})du \;= \frac{8}{\pi^2}\int_{-\infty}^0 \frac{u^{2n+1}}{e^{2u}-1}e^{u}du$$

Variables logarítmicas

$$= \frac{8}{\pi^2}\int_{-\infty}^{0} u^{2n+1}\frac{e^{u}}{e^{2u}-1}du \quad = \frac{8}{\pi^2}\int_{-\infty}^{0} u^{2n+1}\left[-\sum_{k=0}^{\infty}(e^{u})^{2k+1}\right]du$$

$$= -\frac{8}{\pi^2}\int_{-\infty}^{0}\sum_{k=0}^{\infty}u^{2n+1}e^{u(2k+1)}du \quad = -\frac{8}{\pi^2}\sum_{k=0}^{\infty}\int_{-\infty}^{0}u^{2n+1}e^{u(2k+1)}du$$

Tomando $u = \dfrac{y}{2k+1} \quad \Rightarrow \quad du = \dfrac{1}{2k+1}dy$

$$\Rightarrow \quad \mathbb{E}[(\Lambda)^{2n}] = -\frac{8}{\pi^2}\sum_{k=0}^{\infty}\int_{-\infty}^{0}\left(\frac{y}{2k+1}\right)^{2n+1}e^{\frac{y}{2k+1}(2k+1)}\frac{1}{2k+1}du$$

$$= -\frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n+2}}\int_{-\infty}^{0}y^{2n+1}e^{y}du$$

Variables logarítmicas

Tomando $y = -t \quad \Rightarrow \quad dy = -dt$

$$\Rightarrow \quad \mathbb{E}[(\Lambda)^{2n}] = -\frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n+2}}(-1)^{2n+1}\int_0^{\infty} t^{2n+1}e^{-t}dt$$

$$= \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n+2}}\Gamma(2n+2)$$

Por otro lado,

$$\zeta(2n+2) = \sum_{k=1}^{\infty}\frac{1}{(k)^{2n+2}} \quad = \sum_{k=1}^{\infty}\frac{1}{(2k)^{2n+2}} + \sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n+2}}$$

$$= 2^{-2n-2}\zeta(2n+2) + \sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n+2}}$$

$$\Rightarrow \sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n+2}} = \zeta(2n+2) - 2^{-2n-2}\zeta(2n+2) \quad = \zeta(2n+2)\left(1-2^{-2n-2}\right)$$

Variables logarítmicas

Con esto demostramos que:

Teorema 2

Si $\Lambda := \log\left|\mathbb{C}_1\mathbb{C}_2\right|$, entonces

$$\mathbb{E}[(\Lambda)^{2n}] = \frac{8}{\pi^2}\zeta(2n+2)\left(1-2^{-2n-2}\right)\Gamma(2n+2)$$

Ahora, calcularemos de otra manera el valor de $\mathbb{E}[(\Lambda)^{2n}]$

Para ello sera necesario el siguiente resultado:

Proposición 3

Sea $\tilde{C} := \frac{2}{\pi}\log\left|\mathbb{C}_1\mathbb{C}_2\right|$, entonces

$$\mathbb{E}[e^{\theta\tilde{C}}] = \sec^2(\theta)$$

Variables logarítmicas

Demostración. –

Tenemos que $\mathbb{E}[e^{\theta \tilde{C}}] = \mathbb{E}\left[\exp\left(\theta \frac{2}{\pi}\log\left|\mathbb{C}_1\mathbb{C}_2\right|\right)\right] = \mathbb{E}\left[\left|\mathbb{C}_1\mathbb{C}_2\right|^{\theta(2/\pi)}\right]$

$= \mathbb{E}\left[\left|\mathbb{C}_1\right|^{\theta(2/\pi)}\right]\mathbb{E}\left[\left|\mathbb{C}_2\right|^{\theta(2/\pi)}\right] = \mathbb{E}^2\left[\left|\mathbb{C}_1\right|^{\theta(2/\pi)}\right] = \mathbb{E}^2\left[\left(\mathbb{C}_1{}^2\right)^{\theta/\pi}\right]$

Pero recordemos que $\mathbb{C}_1 \sim N/N'$, donde son dos variables normales estandar

y $N^2 \sim \chi_1^2$

$$\Rightarrow \mathbb{E}[e^{\theta \tilde{C}}] = \mathbb{E}^2\left[\left(\frac{N^2}{N'^2}\right)^{\theta/\pi}\right] = \mathbb{E}^2\left[\left(\frac{\chi_1^2}{\chi'^2_1}\right)^{\theta/\pi}\right] = \mathbb{E}^2\left[\left(\chi_1^2\right)^{\theta/\pi}\right]\mathbb{E}^2\left[\left(\chi'^2_1\right)^{-\theta/\pi}\right]$$

$$= \left(\int_0^\infty x^{\theta/\pi}\chi(x)dx\right)^2\left(\int_0^\infty x^{-\theta/\pi}\chi(x)dx\right)^2$$

Variables logarítmicas

$$= \left(\int_0^\infty x^{\theta/\pi} \frac{x^{-1/2} e^{-x/2}}{\sqrt{2} \cdot \Gamma(\frac{1}{2})} dx \right)^2 \left(\int_0^\infty x^{-\theta/\pi} \frac{x^{-1/2} e^{-x/2}}{\sqrt{2} \cdot \Gamma(\frac{1}{2})} dx \right)^2$$

$$= \frac{1}{4\pi^2} \left(\int_0^\infty x^{\theta/\pi - 1/2} e^{-x/2} dx \right)^2 \left(\int_0^\infty x^{-\theta/\pi - 1/2} e^{-x/2} dx \right)^2$$

Sea $x = 2t \quad \Rightarrow \quad dx = 2dt$

$$\Rightarrow \quad \mathbb{E}[e^{\theta \tilde{C}}] = \frac{1}{4\pi^2} \left(\int_0^\infty 2^{\theta/\pi - 1/2} t^{\theta/\pi - 1/2} e^{-t} 2dt \right)^2 \left(\int_0^\infty 2^{-\theta/\pi - 1/2} t^{-\theta/\pi - 1/2} e^{-t} 2dt \right)^2$$

$$= \frac{1}{\pi^2} \left(\int_0^\infty t^{\theta/\pi - 1/2} e^{-t} dt \right)^2 \left(\int_0^\infty t^{-\theta/\pi - 1/2} e^{-t} dt \right)^2$$

$$= \frac{1}{\pi^2} \Gamma^2 \left(\frac{\theta}{\pi} + \frac{1}{2} \right) \Gamma^2 \left(-\frac{\theta}{\pi} + \frac{1}{2} \right)$$

Variables logarítmicas

y aqui hacemos uso de la formula de reflexion de Euler:

Lema 1 (formula de reflexión de Euler)

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad \forall z \notin \mathbb{Z}$$

$$\Rightarrow \quad \mathbb{E}[e^{\theta \tilde{C}}] = \left[\frac{1}{\pi}\Gamma\left(\frac{\theta}{\pi}+\frac{1}{2}\right)\Gamma\left(-\frac{\theta}{\pi}+\frac{1}{2}\right)\right]^2 \quad = \left[\frac{1}{\sin\left(\pi\left(\frac{\theta}{\pi}+\frac{1}{2}\right)\right)}\right]^2 = \frac{1}{\sin^2\left(\theta+\frac{\pi}{2}\right)} = \frac{1}{\cos^2(\theta)}$$

$$\therefore \quad \mathbb{E}[e^{\theta \tilde{C}}] = \sec^2(\theta), \qquad \tilde{C} = \frac{2}{\pi}\log\left|\mathbb{C}_1\mathbb{C}_2\right|$$

Variables logarítmicas

Ahora, recordemos que nuestro objetivo es calcular $\mathbb{E}[(\Lambda)^{2n}]$

$$\Rightarrow \quad \mathbb{E}[\Lambda^{2n}] = \mathbb{E}[(\frac{\pi}{2}\tilde{C})^{2n}] = \left(\frac{\pi}{2}\right)^{2n} \mathbb{E}[\tilde{C}^{2n}]$$

Consideremos la funcion generadora de momentos:

$$\mathbb{E}[e^{t\cdot\tilde{C}}] = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbb{E}[\tilde{C}^k]$$

pero $\mathbb{E}[e^{t\cdot\tilde{C}}] = \sec^2(t) = \sum_{n=0}^{\infty} \frac{L_n}{n!} t^n$

$$\Rightarrow \quad \frac{t^{2n}}{(2n)!}\mathbb{E}[\tilde{C}^{2n}] = \frac{L_{2n}}{(2n)!} t^{2n} \quad \Rightarrow \quad \mathbb{E}[\tilde{C}^{2n}] = L_{2n}$$

Variables logarítmicas

De donde

Proposición 4

$$\mathbb{E}[\Lambda^{2n}] = \left(\frac{\pi}{2}\right)^{2n} L_{2n}$$

Y juntando esto con el Teorema 2, obenemos

$$\begin{cases} \mathbb{E}[(\Lambda)^{2n}] = \dfrac{8}{\pi^2}\zeta(2n+2)\left(1-2^{-2n-2}\right)\Gamma(2n+2) \\ \mathbb{E}[(\Lambda)^{2n}] = \left(\dfrac{\pi}{2}\right)^{2n} L_{2n} \end{cases}$$

Variables logarítmicas

Teorema 3

$$\left(1-2^{-2n-2}\right)\zeta(2n+2)=\frac{1}{2}\left(\frac{\pi}{2}\right)^{2n+2}\frac{L_{2n}}{\Gamma(2n+2)}$$

¿Quien es L_{2n} ?

Numeros de Euler, numeros secante, etc...

Sabemos que $\tan(x)=\cot(x)-2\cot(2x) \;\Rightarrow\; \tan(ix)=\cot(ix)-2\cot(2ix)$

$$\Rightarrow\; \tanh(x)=2\coth(2x)-\coth(x)$$

$$\Rightarrow\; \tanh(x)=2\left(\sum_{n=0}^{\infty}\frac{B_{2n}2^{2n}}{(2n)!}(2x)^{2n-1}\right)-\left(\sum_{n=0}^{\infty}\frac{B_{2n}2^{2n}}{(2n)!}x^{2n-1}\right)$$

$$\Rightarrow\; \tanh(x)=\sum_{n=0}^{\infty}\frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}x^{2n-1}$$

Variables logarítmicas

y como $\dfrac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$

$$\Rightarrow \quad \operatorname{sech}^2(x) = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)B_{2n}}{(2n)!} x^{2n-2}$$

$$\Rightarrow \quad \sec^2(x) = \operatorname{sech}^2(ix) = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)B_{2n}}{(2n)!} i^{2n-2} x^{2n-2}$$

$$\Rightarrow \quad \sec^2(x) = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)B_{2n}}{(2n)!} (-1)^n x^{2n-2}$$

$$\therefore \quad L_{2n} = 2^{2n+2}(2^{2n+2}-1)(2n+1)B_{2n+2}(-1)^{n+1}$$

$$\Rightarrow \quad \zeta(2n) = \frac{(-1)^{n-1} 2^{2n-1} B_{2n}}{(2n)!} \pi^{2n} \quad \forall n \geq 1$$

Extra

¿ Que pasa con $\zeta(2n+1), n \geq 1$?

- Euler conjeturo que $\zeta(2n+1) = \frac{a}{b}\pi^{2n+1}$, sin embargo no pudo demostrarlo
- Euler en 1772 encontro una "formula de recurrencia" para $\zeta(3)$

$$\zeta(3) = \frac{\pi^2}{7}\left(1 - 4\sum_{n=1}^{\infty}\frac{\zeta(2k)}{2^{2k}(2k+1)(2k+2)}\right)$$

- A lo largo de los años se encontraron otras representaciones para ζ en impares
- En el año 1978, Roger Apéry en apoyo de Henri Cohen, Hendrik Lenstra y Alfred van der Poorten, dio una prueba de que $\zeta(3) \approx 1.20205....$ es un numero irracional. Su prueba no se pudo extender a otros valores
- En el año 2000, Tanguy Rivoal demostro que hay una infinidad de $n \in \mathbb{N}$ tales que $\zeta(2n+1)$ es irracional

Extra

- En el año 2001, V. Zudilin, demostro almenos uno de los cuatro numeros

$\zeta(5)$, $\zeta(7)$, $\zeta(9)$,$\zeta(11)$ es irracional

- En el año 2002, Cvijovic y Klinowski, demostraron la siguiente representacion integral para los impares

$$\zeta(2n+1) = \frac{(-1)^{n-1}2^n}{(2n+1)!}\pi^{2n+1}\int_0^1 B_{2n+1}(x)\cot(\pi x)dx$$

- Mas aun, se tiene que

$$\sum_{n=2}^{\infty}\left\{\zeta(n)\right\} = 1$$

y ademas

$$\sum_{n=1}^{\infty}\left\{\zeta(2n)\right\} = \frac{3}{4} \quad \text{y} \quad \sum_{n=1}^{\infty}\left\{\zeta(2n+1)\right\} = \frac{1}{4}$$

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