Universidad Nacional Autónoma de México

Facultad de Ciencias

Variable Compleja III

Tarea 2

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Problema 1. –

vale(2.0) Para $w \in \mathbb{D}$ y $z \in \mathbb{D}$, en núcleo de Poisson se define

$$K(w,z) = \frac{1-|z|^2}{|w-z|^2}$$

Demuestre que

$$K(e^{it}, z) = \mathrm{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right)$$

Demostración: En efecto, sea $w = e^{it}$ $(w \in \overline{D})$ con $t \in \mathbb{R}$ y $z \in D$, entonces

$$\mathrm{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right) = \mathrm{Re}\left(\frac{e^{it}+z}{e^{it}-z}\cdot\frac{e^{-it}-\overline{z}}{e^{-it}-\overline{z}}\right) = \mathrm{Re}\left(\frac{1-|z|^2}{\left|e^{it}-z\right|^2}\right) = \frac{1-|z|^2}{\left|e^{it}-z\right|^2} = K(e^{it}, z)$$

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Problema 2. –

vale(2.0) Demuestre que la ecuación de Laplace es invariante bajo cambio de coordenadas analíticos. Es decir, sea la ecuación de Laplace:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0, \tag{1}$$

donde $w : \mathbb{R}^2 \to \mathbb{R}$. Si consideramos el cambio de variables independientes $(x,y) \to (u,v)$ donde

$$f(x,y) = u(x,y) + iv(x,y)$$

es analítica (holomorfa) e invertible entonces, en las nuevas variables independientes, la ecuación (1) se transforma en:

$$\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2} = 0$$

Demostración: Tenemos que

$$0 = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial x}\left[\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[\frac{\partial w}{\partial y}\right] = \frac{\partial}{\partial x}\left[\frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial y}\left[\frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial v}\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial u}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial v}\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial w}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial v}\right)\frac{\partial v}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial^2 v}{\partial x^2} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial w}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial v}\right)\frac{\partial v}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial^2 v}{\partial y^2}$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial v}\right)\frac{\partial v}{\partial x} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial v}\right)\frac{\partial v}{\partial y} + \left[\frac{\partial w}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial v}\frac{\partial^2 v}{\partial x^2} + \frac{\partial w}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial w}{\partial v}\frac{\partial^2 v}{\partial y^2}\right]$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial v}\right)\frac{\partial v}{\partial x} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial v}\right)\frac{\partial v}{\partial y} + \left[\frac{\partial w}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial v}\frac{\partial^2 v}{\partial x^2} - \frac{\partial w}{\partial u}\frac{\partial^2 u}{\partial x^2} - \frac{\partial w}{\partial v}\frac{\partial^2 v}{\partial x^2}\right]$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial v}\right)\frac{\partial v}{\partial x} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial v}\right)\frac{\partial v}{\partial y}$$
$$= \left[\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v \partial u}\frac{\partial v}{\partial x}\right]\frac{\partial u}{\partial x} + \left(\frac{\partial w}{\partial u \partial v}\frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial x} + \left(\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v \partial u}\frac{\partial v}{\partial y}\right)\frac{\partial u}{\partial y} + \left(\frac{\partial w}{\partial u \partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial y}$$
$$= \left[\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v \partial u}\frac{\partial v}{\partial x}\right]\frac{\partial u}{\partial x} + \left(\frac{\partial w}{\partial u \partial v}\frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial x} + \left(-\frac{\partial^2 w}{\partial u^2}\frac{\partial v}{\partial y} - \frac{\partial w}{\partial v \partial u}\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} + \left(\frac{\partial w}{\partial u \partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v^2}\frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial y}$$
$$= \left(\frac{\partial u}{\partial x}\right)^2\left[\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2}\right] + \left(\frac{\partial v}{\partial y}\right)^2\left[\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2}\right] = \left[\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2}\right]\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

y como necesariamente u, v no son constantes al mismo tiempo (pues es un cambio de variable) tenemos que

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \neq 0$$

por lo que

$$\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2} = 0$$

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Problema 3. –

vale(2.0) Sea $\Omega \subseteq \mathbb{R}^2$ una región y u función armónica en Ω. Si u se anula en al menos un disco $D(z_0, \epsilon) \subset \Omega$, demuestre que $u = 0$ en todo Ω.

Demostración: Consideremos la función $f : \Omega \to \mathbb{C}$ dada por $f(x+iy) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$, la cual es analítica en Ω pues $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ son C^1 y cumplen C-R pues

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} \underset{\text{armonica}}{=} -\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)$$
$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} \underset{C^1}{=} \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right)$$

Con esto notemos que si $x + iy \in D(z_0, \varepsilon)$ entonces

$$\frac{\partial u}{\partial x}(x,y) = \lim_{h \to 0}\frac{u(x+h,y) - u(x,y)}{h} = \lim_{\substack{h \to 0 \\ h < \varepsilon}}\frac{u(x+h,y)}{h} = \lim_{h \to 0}\frac{0}{h} = 0$$

$$\frac{\partial u}{\partial y}(x,y)=\lim_{h\to 0}\frac{u(x,y+h)-u(x,y)}{h}=\lim_{\substack{h\to 0\\ h<\varepsilon}}\frac{u(x,y+h)}{h}=\lim_{h\to 0}\frac{0}{h}=0$$

por lo tanto $f(z)=0,\ \forall z\in D(z_0,\varepsilon)$, con esto y dado que la función idénticamente cero también es analítica en Ω y 0 en $D(z_0,\varepsilon)$ tenemos por continuación analítica que $f(z)=0,\ \forall z\in\Omega$, por lo que

$$\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}=0 \iff \frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=0 \iff u\equiv 0$$

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Problema 4. –
vale(2.0) Sea $h:\partial\mathbb{D}\to\mathbb{R}$ na función continua. Demuestre que

$$u(z)=\frac{1}{2\pi}\int_0^{2\pi} h(e^{ti})\frac{1-|z|^2}{|e^{ti}-z|^2}\,dt$$

es armónica para todo $z\in\mathbb{D}$

Demostración: Notemos que por el problema 1 (y por el hecho de que $u(e^{it})$ está bien definido)

$$u(z)=\frac{1}{2\pi}\int_0^{2\pi}h(e^{it})\frac{1-|z|^2}{|e^{it}-z|^2}dt=\frac{1}{2\pi}\int_0^{2\pi}h(e^{it})\operatorname{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right)dt=\operatorname{Re}\left(\frac{1}{2\pi}\int_0^{2\pi}h(e^{it})\frac{e^{it}+z}{e^{it}-z}dt\right)$$

entonces sea

$$g:D\to\mathbb{C},\ \ g(z)=\frac{1}{2\pi}\int_0^{2\pi}h(e^{it})\frac{e^{it}+z}{e^{it}-z}dt$$

y basta probar que esta función es analítica. Sea $f:[0,2\pi]\times D\to\mathbb{C}$ dada por

$$f(t,z)=h(e^{it})\frac{e^{it}+z}{e^{it}-z}\ \ \Rightarrow\ \ g(z)=\frac{1}{2\pi}\int_0^{2\pi}f(t,z)dt$$

tenemos las siguientes

- Para cada $t\in[0,2\pi]$, $f(t,z)$ es continua, pues al ser $z\in D$, $e^{it}-z\neq 0$.
- Tenemos que

$$\frac{\partial f}{\partial z}(t,z)=h(e^{it})\frac{2e^{it}}{(e^{it}-z)^2}$$

la cual nuevamente para cada $t\in[0,2\pi]$ es continua.

Por lo tanto, por la regla integral de Leibniz tenemos que $g(z)$ es analítica, y por tanto, u es armónica.

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Problema 5. –
vale(2.0) Sea u armónica en $\mathbb{C} \setminus \{0\}$. Como el conjunto no es simplemente conexo, no podemos asegurar la exitencia de una armónica conjugada. Demuestre que el logaritmo natural es la única obstrucción para la existencia de una conjugada. Esto es, demuestre que existe una constante c tal que

$$\tilde{u}(x, y) = u(x, y) - c \ln(x^2 + y^2)$$

es armónica y existe para ella una conjugada armónica $\tilde{v}$.

Demostración: