

EULER'S FORMULAE FOR $\zeta(2n)$ AND PRODUCTS OF CAUCHY VARIABLES

Lorenzo Antonio Alvarado Cabrera

26/febrero/2026

Contenido

- Números de Bernoulli
- Un vistazo rápido a la prueba de Euler
- Prueba usando variables logarítmicas de Cauchy

Números de Bernoulli

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$$

⋮

$$1^m + 2^m + 3^m + \dots + n^m = \text{¿?} \quad \text{con } m \in \mathbb{Z}^+$$

Llamemos $S_m(n) = 1^m + 2^m + \dots + (n-1)^m$, $n \geq 1$, $m \geq 0$

$$\Rightarrow S_{m+1}(n+1) = 1 + \sum_{i=0}^{m+1} \binom{m+1}{i} S_i(n)$$

$$\Rightarrow S_m(n) = \frac{1}{m+1} \left[n^{m+1} - 1 - \sum_{i=0}^{m-1} \binom{m+1}{i} S_i(n) \right]$$

Números de Bernoulli

$$S_1(n) = \frac{1}{2}n^2 - \frac{1}{2}n$$

$$S_2(n) = \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$$

$$S_3(n) = \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2$$

$$S_4(n) = \frac{1}{5}n^5 - \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

$$S_5(n) = \frac{1}{6}n^6 - \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$$

Números de Bernoulli

- $$S_1(n) = \frac{1}{2} \left[\binom{2}{0} n^2 + \binom{2}{1} \left(-\frac{1}{2}\right) n \right]$$
- $$S_2(n) = \frac{1}{3} \left[\binom{3}{0} n^3 + \binom{3}{1} \left(-\frac{1}{2}\right) n^2 + \binom{3}{2} \left(\frac{1}{6}\right) n \right]$$
- $$S_3(n) = \frac{1}{4} \left[\binom{4}{0} n^4 + \binom{4}{1} \left(-\frac{1}{2}\right) n^3 + \binom{4}{2} \left(\frac{1}{6}\right) n^2 + \binom{4}{3} \cdot 0 \cdot n \right]$$
- $$S_4(n) = \frac{1}{5} \left[\binom{5}{0} n^5 + \binom{5}{1} \left(-\frac{1}{2}\right) n^4 + \binom{5}{2} \left(\frac{1}{6}\right) n^3 + \binom{5}{3} \cdot 0 \cdot n^2 + \binom{5}{4} \left(-\frac{1}{30}\right) n \right]$$
- $$S_5(n) = \frac{1}{6} \left[\binom{6}{0} n^6 + \binom{6}{1} \left(-\frac{1}{2}\right) n^5 + \binom{6}{2} \left(\frac{1}{6}\right) n^4 + \binom{6}{3} \cdot 0 \cdot n^3 + \binom{6}{4} \left(-\frac{1}{30}\right) n^2 + \binom{6}{5} \cdot 0 \cdot n \right]$$

Números de Bernoulli

$$\text{Pero } S_m(n) = \frac{1}{m+1} \left[n^{m+1} - 1 - \sum_{i=0}^{m-1} \binom{m+1}{i} S_i(n) \right]$$
$$\Rightarrow S'_m(0) = -\frac{1}{m+1} \sum_{i=0}^{m-1} \binom{m+1}{i} S'_i(0)$$

Definición

Se definen a los **Numeros de Bernoulli** como:

$$B_0 = 1, \quad B_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k$$

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0$$

Números de Bernoulli

∴ se obtiene la formula para $S_m(n+1)$

$$S_m(n+1) = 1^m + 2^m + \dots + n^m = \frac{1}{m+1} \sum_{j=0}^m \binom{m+1}{j} B_j n^{m+1-j}$$

$$= \frac{1}{m+1} \left[n^{m+1} + \binom{m+1}{1} B_1 n^m + \binom{m+1}{2} B_2 n^{m-1} + \dots + \binom{m+1}{m} B_m n \right]$$

Números de Bernoulli

Proposición 1

Los números de Bernoulli cumplen:

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = \frac{z}{e^z - 1}, \quad \forall |z| < \pi$$

Proposición 2

Se tiene que $B_{2n+1} = 0 \quad \forall n \geq 1$

Demostración. –

Consideremos a la función $\coth(z) := \frac{e^z + e^{-z}}{e^z - e^{-z}}$

$$\Rightarrow \coth(z) - 1 = \frac{2e^{-z}}{e^z - e^{-z}} = \frac{2}{e^{2z} - 1} \quad \Rightarrow \quad \frac{z}{2} \coth\left(\frac{z}{2}\right) = \frac{z}{2} + \frac{z}{e^z - 1}$$

Números de Bernoulli

$$\Rightarrow \frac{z}{2} \coth\left(\frac{z}{2}\right) = \frac{z}{2} + \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = 1 + \sum_{n=2}^{\infty} \frac{B_n}{n!} z^n$$

pero el lado izquierdo es una función par, por lo que $B_{2n+1} = 0$

Teorema 1

$$\coth(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{B_{2n} 2^{2n}}{(2n)!} z^{2n-1}, \quad \forall |z| < \pi$$

Prueba de Euler

Consideremos

$$\sin(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2} \right)$$

$$\Rightarrow -i \sin(iz) = -i(iz) \prod_{n=1}^{\infty} \left(1 - \frac{(iz)^2}{\pi^2 n^2} \right) = z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{\pi^2 n^2} \right)$$

↓

$$\sinh(z)$$

$$\Rightarrow \frac{\sinh(z)}{z} = \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{\pi^2 n^2} \right)$$

$$\Rightarrow \log\left(\frac{\sinh(z)}{z}\right) = \log \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{\pi^2 n^2} \right) = \sum_{n=1}^{\infty} \log \left(1 + \frac{z^2}{\pi^2 n^2} \right)$$



Prueba de Euler

Derivando de ambos lados obtenemos que

$$\begin{aligned}\Rightarrow \coth(z) - \frac{1}{z} &= 2 \sum_{n=1}^{\infty} \frac{z}{z^2 + \pi^2 n^2} = 2 \sum_{n=1}^{\infty} \frac{z}{\pi^2 n^2} \frac{1}{(z/\pi n)^2 + 1} \\ &= 2 \sum_{n=1}^{\infty} \frac{z}{\pi^2 n^2} \sum_{m=1}^{\infty} (-1)^{m-1} \left(\frac{z}{\pi n}\right)^{2m-2} \\ \Rightarrow \coth(z) - \frac{1}{z} &= 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{m-1} \frac{z^{2m-1}}{\pi^{2m} n^{2m}} = 2 \sum_{m=1}^{\infty} (-1)^{m-1} \frac{z^{2m-1}}{\pi^{2m}} \sum_{n=1}^{\infty} \frac{1}{n^{2m}} \\ &= \sum_{m=1}^{\infty} (-1)^{m-1} \frac{2z^{2m-1}}{\pi^{2m}} \zeta(2m)\end{aligned}$$

Prueba de Euler

$$\therefore \coth(z) = \frac{1}{z} + \sum_{n=1}^{\infty} 2(-1)^{n-1} \frac{z^{2n-1}}{\pi^{2n}} \zeta(2n), \quad \forall |z| < \pi$$

pero por el Teo 1

$$\coth(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{B_{2n} 2^{2n}}{(2n)!} z^{2n-1}, \quad \forall |z| < \pi$$

$$\Rightarrow 2(-1)^{n-1} \frac{1}{\pi^{2n}} \zeta(2n) = \frac{B_{2n} 2^{2n}}{(2n)!} \quad \Rightarrow \quad \zeta(2n) = \frac{(-1)^{n-1} 2^{2n-1} B_{2n} \pi^{2n}}{(2n)!} \quad \forall n \geq 1$$

Variables logarítmicas

Consideremos dos variables de Cauchy \mathbb{C}_1 y \mathbb{C}_2 independientes y sea:

$$\Lambda := \log |\mathbb{C}_1 \mathbb{C}_2|$$

Sabemos que la función de densidad de $\mathbb{C}_1 \mathbb{C}_2$, viene dada por:

$$\psi(x) = \frac{2 \log |x|}{\pi^2 (x^2 - 1)}$$

Entonces la esperanza de $(\Lambda)^{2n}$ vendrá dada por

$$\begin{aligned} \mathbb{E}[(\Lambda)^{2n}] &= \int_{-\infty}^{\infty} (\log |x|)^{2n} \psi(x) dx = \int_{-\infty}^{\infty} (\log |x|)^{2n} \frac{2 \log |x|}{\pi^2 (x^2 - 1)} dx \\ &= 2 \int_0^{\infty} \frac{2 \log^{2n+1}(x)}{\pi^2 (x^2 - 1)} dx = \frac{4}{\pi^2} \int_0^{\infty} \frac{\log^{2n+1}(x)}{x^2 - 1} dx \end{aligned}$$

Variables logarítmicas

$$= \frac{4}{\pi^2} \int_0^1 \frac{\log^{2n+1}(x)}{x^2 - 1} dx + \frac{4}{\pi^2} \int_1^\infty \frac{\log^{2n+1}(x)}{x^2 - 1} dx$$

Tomando $x = \frac{1}{y}$ en la primer integral $\Rightarrow dx = -\frac{1}{y^2} dy$

$$\int_0^1 \frac{\log^{2n+1}(x)}{x^2 - 1} dx = \int_\infty^1 \frac{\log^{2n+1}(1/y)}{(1/y)^2 - 1} \left(-\frac{1}{y^2}\right) dy = \int_1^\infty \frac{\log^{2n+1}(y)}{y^2 - 1} dy$$

$$\Rightarrow \mathbb{E}[(\Lambda)^{2n}] = \frac{8}{\pi^2} \int_1^\infty \frac{\log^{2n+1}(x)}{x^2 - 1} dx$$

Tomando $x = e^{-u}$ $\Rightarrow dx = -e^{-u} du$

$$\Rightarrow \mathbb{E}[(\Lambda)^{2n}] = \frac{8}{\pi^2} \int_0^{-\infty} \frac{\log^{2n+1}(e^{-u})}{(e^{-u})^2 - 1} (-e^{-u}) du = \frac{8}{\pi^2} \int_{-\infty}^0 \frac{u^{2n+1}}{e^{2u} - 1} e^u du$$

Variables logarítmicas

$$\begin{aligned} &= \frac{8}{\pi^2} \int_{-\infty}^0 u^{2n+1} \frac{e^u}{e^{2u} - 1} du = \frac{8}{\pi^2} \int_{-\infty}^0 u^{2n+1} \left[-\sum_{k=0}^{\infty} (e^u)^{2k+1} \right] du \\ &= -\frac{8}{\pi^2} \int_{-\infty}^0 \sum_{k=0}^{\infty} u^{2n+1} e^{u(2k+1)} du = -\frac{8}{\pi^2} \sum_{k=0}^{\infty} \int_{-\infty}^0 u^{2n+1} e^{u(2k+1)} du \end{aligned}$$

Tomando $u = \frac{y}{2k+1} \Rightarrow du = \frac{1}{2k+1} dy$

$$\begin{aligned} \Rightarrow \mathbb{E}[(\Lambda)^{2n}] &= -\frac{8}{\pi^2} \sum_{k=0}^{\infty} \int_{-\infty}^0 \left(\frac{y}{2k+1}\right)^{2n+1} e^{\frac{y}{2k+1}(2k+1)} \frac{1}{2k+1} dy \\ &= -\frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n+2}} \int_{-\infty}^0 y^{2n+1} e^y dy \end{aligned}$$

Variables logarítmicas

Tomando $y = -t \Rightarrow dy = -dt$

$$\begin{aligned}\Rightarrow \mathbb{E}[(\Lambda)^{2n}] &= -\frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n+2}} (-1)^{2n+1} \int_0^{\infty} t^{2n+1} e^{-t} dt \\ &= \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n+2}} \Gamma(2n+2)\end{aligned}$$

Por otro lado,

$$\begin{aligned}\zeta(2n+2) &= \sum_{k=1}^{\infty} \frac{1}{k^{2n+2}} = \sum_{k=1}^{\infty} \frac{1}{(2k)^{2n+2}} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n+2}} \\ &= 2^{-2n-2} \zeta(2n+2) + \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n+2}}\end{aligned}$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n+2}} = \zeta(2n+2) - 2^{-2n-2} \zeta(2n+2) = \zeta(2n+2) (1 - 2^{-2n-2})$$

Variables logarítmicas

Con esto demostramos que:

Teorema 2

Si $\Lambda := \log|\mathbb{C}_1\mathbb{C}_2|$, entonces

$$\mathbb{E}[(\Lambda)^{2n}] = \frac{8}{\pi^2} \zeta(2n+2) (1 - 2^{-2n-2}) \Gamma(2n+2)$$

Ahora, calcularemos de otra manera el valor de $\mathbb{E}[(\Lambda)^{2n}]$

Para ello sera necesario el siguiente resultado:

Proposición 3

Sea $\tilde{C} := \frac{2}{\pi} \log|\mathbb{C}_1\mathbb{C}_2|$, entonces

$$\mathbb{E}[e^{\theta\tilde{C}}] = \sec^2(\theta), \quad |\theta| < \frac{\pi}{2}$$

Variables logarítmicas

Demostración. –

$$\begin{aligned}\text{Tenemos que } \mathbb{E}[e^{\theta\tilde{C}}] &= \mathbb{E}\left[\exp\left(\theta\frac{2}{\pi}\log|\mathbb{C}_1\mathbb{C}_2|\right)\right] = \mathbb{E}\left[|\mathbb{C}_1\mathbb{C}_2|^{\theta(2/\pi)}\right] \\ &= \mathbb{E}\left[|\mathbb{C}_1|^{\theta(2/\pi)}\right]\mathbb{E}\left[|\mathbb{C}_2|^{\theta(2/\pi)}\right] = \mathbb{E}^2\left[|\mathbb{C}_1|^{\theta(2/\pi)}\right] = \mathbb{E}^2\left[\left(\mathbb{C}_1^2\right)^{\theta/\pi}\right]\end{aligned}$$

Pero recordemos que $\mathbb{C}_1 \sim N/N'$, donde son dos variables normales estandar independientes y $N^2 \sim \chi_1^2$

$$\begin{aligned}\Rightarrow \mathbb{E}[e^{\theta\tilde{C}}] &= \mathbb{E}^2\left[\left(\frac{N^2}{N'^2}\right)^{\theta/\pi}\right] = \mathbb{E}^2\left[\left(\frac{\chi_1^2}{\chi_1'^2}\right)^{\theta/\pi}\right] = \mathbb{E}^2\left[\left(\chi_1^2\right)^{\theta/\pi}\right]\mathbb{E}^2\left[\left(\chi_1'^2\right)^{-\theta/\pi}\right] \\ &= \left(\int_0^\infty x^{\theta/\pi}\chi(x)dx\right)^2 \left(\int_0^\infty x^{-\theta/\pi}\chi(x)dx\right)^2\end{aligned}$$

Variables logarítmicas

$$\begin{aligned} &= \left(\int_0^\infty x^{\theta/\pi} \frac{x^{-1/2} e^{-x/2}}{\sqrt{2} \cdot \Gamma(\frac{1}{2})} dx \right)^2 \left(\int_0^\infty x^{-\theta/\pi} \frac{x^{-1/2} e^{-x/2}}{\sqrt{2} \cdot \Gamma(\frac{1}{2})} dx \right)^2 \\ &= \frac{1}{4\pi^2} \left(\int_0^\infty x^{\theta/\pi-1/2} e^{-x/2} dx \right)^2 \left(\int_0^\infty x^{-\theta/\pi-1/2} e^{-x/2} dx \right)^2 \end{aligned}$$

Sea $x = 2t \Rightarrow dx = 2dt$

$$\begin{aligned} \Rightarrow \mathbb{E}[e^{\theta\tilde{C}}] &= \frac{1}{4\pi^2} \left(\int_0^\infty 2^{\theta/\pi-1/2} t^{\theta/\pi-1/2} e^{-t} 2dt \right)^2 \left(\int_0^\infty 2^{-\theta/\pi-1/2} t^{-\theta/\pi-1/2} e^{-t} 2dt \right)^2 \\ &= \frac{1}{\pi^2} \left(\int_0^\infty t^{\theta/\pi-1/2} e^{-t} dt \right)^2 \left(\int_0^\infty t^{-\theta/\pi-1/2} e^{-t} dt \right)^2 \\ &= \frac{1}{\pi^2} \Gamma^2\left(\frac{\theta}{\pi} + \frac{1}{2}\right) \Gamma^2\left(-\frac{\theta}{\pi} + \frac{1}{2}\right) \end{aligned}$$

Variables logarítmicas

y aquí hacemos uso de la formula de reflexion de Euler:

Lema 1 (formula de reflexión de Euler)

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad \forall z \notin \mathbb{Z}$$

$$\Rightarrow \mathbb{E}[e^{\theta\tilde{C}}] = \left[\frac{1}{\pi} \Gamma\left(\frac{\theta}{\pi} + \frac{1}{2}\right) \Gamma\left(-\frac{\theta}{\pi} + \frac{1}{2}\right) \right]^2 = \left[\frac{1}{\sin\left(\pi\left(\frac{\theta}{\pi} + \frac{1}{2}\right)\right)} \right]^2 = \frac{1}{\sin^2\left(\theta + \frac{\pi}{2}\right)} = \frac{1}{\cos^2(\theta)}$$

$$\therefore \mathbb{E}[e^{\theta\tilde{C}}] = \sec^2(\theta), \quad \tilde{C} = \frac{2}{\pi} \log|\mathbb{C}_1\mathbb{C}_2|$$

Variables logarítmicas

Ahora, recordemos que nuestro objetivo es calcular $\mathbb{E}[(\Lambda)^{2n}]$

$$\Rightarrow \mathbb{E}[\Lambda^{2n}] = \mathbb{E}\left[\left(\frac{\pi}{2}\tilde{C}\right)^{2n}\right] = \left(\frac{\pi}{2}\right)^{2n} \mathbb{E}[\tilde{C}^{2n}]$$

Consideremos la función generadora de momentos:

$$\mathbb{E}[e^{t\tilde{C}}] = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbb{E}[\tilde{C}^k]$$

$$\text{pero } \mathbb{E}[e^{t\tilde{C}}] = \sec^2(t) = \sum_{n=0}^{\infty} \frac{L_n}{n!} t^n$$

$$\Rightarrow \frac{t^{2n}}{(2n)!} \mathbb{E}[\tilde{C}^{2n}] = \frac{L_{2n}}{(2n)!} t^{2n} \Rightarrow \mathbb{E}[\tilde{C}^{2n}] = L_{2n}$$

Variables logarítmicas

De donde

Proposición 4

$$\mathbb{E}[\Lambda^{2n}] = \left(\frac{\pi}{2}\right)^{2n} L_{2n}$$

Y juntanto esto con el Teorema 2, obtenemos

$$\begin{cases} \mathbb{E}[(\Lambda)^{2n}] = \frac{8}{\pi^2} \zeta(2n+2) \left(1 - 2^{-2n-2}\right) \Gamma(2n+2) \\ \mathbb{E}[(\Lambda)^{2n}] = \left(\frac{\pi}{2}\right)^{2n} L_{2n} \end{cases}$$

Variables logarítmicas

Teorema 3

$$\left(1 - 2^{-2n-2}\right) \zeta(2n+2) = \frac{1}{2} \left(\frac{\pi}{2}\right)^{2n+2} \frac{L_{2n}}{\Gamma(2n+2)}$$

¿Quién es L_{2n} ?

Numeros de Euler, numeros secante, etc...

$$\text{Sabemos que } \tan(x) = \cot(x) - 2 \cot(2x) \Rightarrow \tan(ix) = \cot(ix) - 2 \cot(2ix)$$

$$\Rightarrow \tanh(x) = 2 \coth(2x) - \coth(x)$$

$$\Rightarrow \tanh(x) = 2 \left(\sum_{n=0}^{\infty} \frac{B_{2n} 2^{2n}}{(2n)!} (2x)^{2n-1} \right) - \left(\sum_{n=0}^{\infty} \frac{B_{2n} 2^{2n}}{(2n)!} x^{2n-1} \right)$$

$$\Rightarrow \tanh(x) = \sum_{n=0}^{\infty} \frac{2^{2n} (2^{2n} - 1) B_{2n}}{(2n)!} x^{2n-1}$$

Variables logarítmicas

y como $\frac{d}{dx} \tanh(x) = \operatorname{sech}^2(x)$

$$\Rightarrow \operatorname{sech}^2(x) = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)B_{2n}}{(2n)!} x^{2n-2}$$

$$\Rightarrow \sec^2(x) = \operatorname{sech}^2(ix) = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)B_{2n}}{(2n)!} i^{2n-2} x^{2n-2}$$

$$\Rightarrow \sec^2(x) = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)B_{2n}}{(2n)!} (-1)^n x^{2n-2}$$

$$\therefore L_{2n} = 2^{2n+2}(2^{2n+2}-1)(2n+1)B_{2n+2}(-1)^{n+1}$$

$$\Rightarrow \zeta(2n) = \frac{(-1)^{n-1} 2^{2n-1} B_{2n}}{(2n)!} \pi^{2n} \quad \forall n \geq 1$$

Extra

¿ Que pasa con $\zeta(2n + 1), n \geq 1$?

- Euler conjeturo que $\zeta(2n + 1) = \frac{a}{b} \pi^{2n+1}$, sin embargo no pudo demostrarlo
- Euler en 1772 encontro una "formula de recurrencia" para $\zeta(3)$

$$\zeta(3) = \frac{\pi^2}{7} \left(1 - 4 \sum_{n=1}^{\infty} \frac{\zeta(2k)}{2^{2k} (2k + 1)(2k + 2)} \right)$$

- A lo largo de los años se encontraron otras representaciones para ζ en impares
- En el año 1978, Roger Apéry en apoyo de Henri Cohen, Hendrik Lenstra y Alfred van der Poorten, dio una prueba de que $\zeta(3) \approx 1.20205\dots$ es un numero irracional. Su prueba no se pudo extender a otros valores
- En el año 2000, Tanguy Rivoal demostro que hay una infinidad de $n \in \mathbb{N}$ tales que $\zeta(2n + 1)$ es irracional

Extra

- En el año 2001, V. Zudilin, demostro almenos uno de los cuatro numeros $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ es irracional
- En el año 2002, Cvijovic y Klinowski, demostraron la siguiente representacion integral para los impares

$$\zeta(2n + 1) = \frac{(-1)^{n-1} 2^n}{(2n + 1)!} \pi^{2n+1} \int_0^1 B_{2n+1}(x) \cot(\pi x) dx$$

- Mas aun, se tiene que

$$\sum_{n=2}^{\infty} \{\zeta(n)\} = 1$$

y ademas

$$\sum_{n=1}^{\infty} \{\zeta(2n)\} = \frac{3}{4} \quad \text{y} \quad \sum_{n=1}^{\infty} \{\zeta(2n + 1)\} = \frac{1}{4}$$

Extra

Consideremos $U \sim \text{Unif}(0,1)$ y definamos la nueva variable

$$K = \left\lfloor \frac{1}{U} \right\rfloor$$

¿Cual es su distribución?

$$\begin{aligned} \mathbb{P}[K = k] &= \mathbb{P}\left(\left\lfloor 1/U \right\rfloor = k\right) = \mathbb{P}\left(k \leq 1/U < k+1\right) = \mathbb{P}\left(\frac{1}{k+1} \leq U < \frac{1}{k}\right) \\ &= \frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)} \end{aligned}$$

$$\therefore \sum_{k=1}^{\infty} \mathbb{P}[K = k] = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$$

Extra

Pero entonces

$$\begin{aligned} 1 &= \sum_{k=1}^{\infty} \mathbb{P}[K = k] = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=2}^{\infty} \frac{1}{k(k-1)} = \sum_{k=2}^{\infty} \frac{1}{k^2 - k} = \sum_{k=2}^{\infty} \frac{1/k^2}{1 - 1/k} \\ &= \sum_{k=2}^{\infty} \sum_{n=2}^{\infty} \left(\frac{1}{k}\right)^n = \sum_{n=2}^{\infty} \sum_{k=2}^{\infty} \frac{1}{k^n} = \sum_{n=2}^{\infty} \zeta(n) - 1 \end{aligned}$$

Ademas es claro que $1 < \zeta(n) < 2$, por lo que $\{\zeta(n)\} = \zeta(n) - 1$

$$\therefore \sum_{n=2}^{\infty} \{\zeta(n)\} = 1$$

Extra

"La suma de las partes fraccionarias de los valores de la función zeta es 1 porque corresponde exactamente a la distribución del entero $\lfloor 1/U \rfloor$ cuando $U \sim \text{Unif}(0,1)$ "

$$\text{Ademas } \sum_{n=1}^{\infty} \{\zeta(2n)\} = \mathbb{E} \left[\frac{K}{K-1} \right] = \frac{3}{4} \Rightarrow \sum_{n=1}^{\infty} \{\zeta(2n+1)\} = \frac{1}{4}$$

"Probabilísticamente $\{\zeta(2n)\}$ concentra la mayor parte de la masa"

EULER'S FORMULAE FOR $\zeta(2n)$ AND PRODUCTS OF CAUCHY VARIABLES

Lorenzo Antonio Alvarado Cabrera

26/febrero/2026